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B.Sc 1 (H) Paper 1

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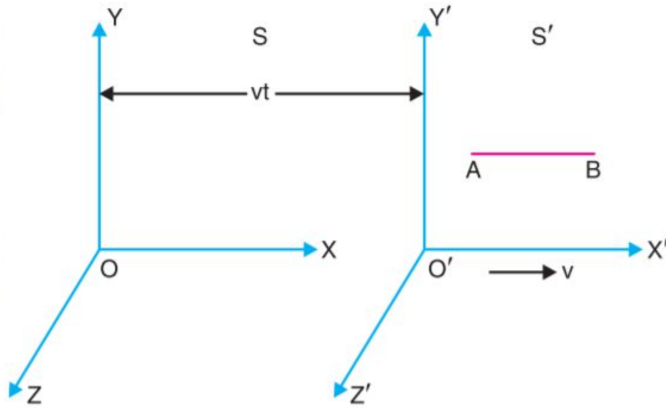
Length Contraction

Consider two coordinate systems S and S' with their X-axes coinciding at time $t = 0$. S' is moving with a uniform relative speed v with respect to S in the positive X-direction. Imagine a rod (AB), at rest relative to S' (Fig. 1.5).



Let x'_1 and x'_2 be the coordinates of the ends of the rod at any instant of time in S'. Then,

$$l_0 = x_2' - x_1' \qquad ...(1)$$

since the rod is at rest in frame S'.

Similarly, let x_1 and x_2 be the coordinates of the ends of the rod at the same instant of time in S.

Then $$l = x_2 - x_1 \qquad ...(2)$$

l the length of the rod, measured relative to S.

According to Lorentz transformations,

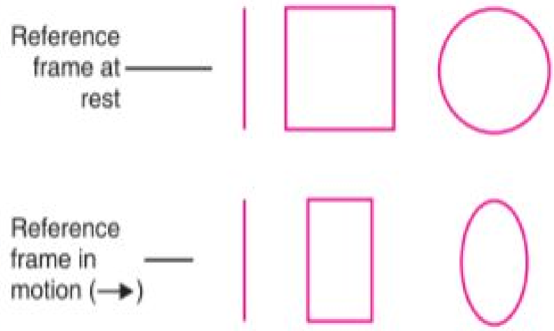
$$x_2' = \frac{x_2 - vt}{\sqrt{1-(v^2/c^2)}}$$

$$x_1' = \frac{x_1 - vt}{\sqrt{1-(v^2/c^2)}}$$

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1-(v^2/c^2)}} \quad \text{or} \quad l_0 = \frac{l}{\sqrt{1-(v^2/c^2)}}$$

$$l = l_0\sqrt{1-(v^2/c^2)}$$

From equation (5) we see that $l < l_0$. Therefore, to the observer in S it would appear that the length of the rod (in S') has contracted by the factor $\sqrt{1-(v^2/c^2)}$.



For example, a body which appears to be spherical to an observer at rest relative to it, will appear to be an oblate spheroid to a moving observer. Similarly, a square and a circle in one appear to the observer in the other to be a rectangle and an ellipse respectively

Time Dilation

Imagine a gun placed at the position (x', y', z') in S'. Suppose it fires two shots at times t_1' and t_2' measured with respect to S'. In S' the clock is at rest relative to the observer. The time interval measured by a clock at rest relative to the observer is called the *proper time interval*. Hence, $t_0 = t_2' - t_1'$ is the time interval between the two shots for the observer in S'. Since the gun is fixed in S', it has a velocity v with respect to S in the direction of the positive X-axis. Let $t = t_2 - t_1$ represent the time interval between the two shots as measured by an observer in S.

From inverse Lorentz transformations we have

$$t_1 = \frac{t_1' + vx'/c^2}{\sqrt{1-(v^2/c^2)}} \quad \text{and} \quad t_2 = \frac{t_2' + vx'/c^2}{\sqrt{1-(v^2/c^2)}}$$

$$\therefore \quad t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1-(v^2/c^2)}} \quad \text{or} \quad t = \frac{t_0}{\sqrt{1-(v^2/c^2)}}$$

$$\text{or} \quad t > t_0.$$

Thus, the time interval, between two events occurring at a given point in the moving frame S' appears to be longer to the observer in the stationary frame S; *i.e.*, a stationary clock measures a longer time interval between events occurring in a moving frame of reference than does a clock in the moving frame. This effect is called *time dilation*.

Mass Energy Equivalence

Derivation. Force is defined as rate of change of momentum *i.e.*,

$$F = \frac{d}{dt}(mv) \quad ...(1)$$

According to the theory of relativity, both mass and velocity are variable. Therefore,

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} \quad ...(2)$$

Let the force F displace the body through a distance dx.

Then, the increase in the kinetic energy (dE_k) of the body is equal to the work done ($F\,dx$).

Hence, $$dE_k = F\,dx = m\frac{dv}{dt}dx + v\frac{dm}{dt}dx$$

or $$dE_k = mv\,dv + v^2\,dm \quad ...(3)$$

According to the law of variation of mass with velocity

$$m = \frac{m_0}{\sqrt{(1-v^2/c^2)}} \quad ...(4)$$

Squaring both sides, $$m^2 = \frac{m_0{}^2}{1-(v^2/c^2)}$$

or $$m^2c^2 = m_0^2c^2 + m^2v^2$$

Differentiating, $$c^2\,2m\,dm = m^2\,2v\,dv + v^2\,2m\,dm$$

or $$c^2\,dm = mv\,dv + v^2\,dm \quad ...(5)$$

From equations (3) and (5), $$dE_k = c^2\,dm \quad ...(6)$$

Thus, a change in K.E. dE_k is directly proportional to a change in mass dm.

When a body is at rest, its velocity is zero, (K.E. = 0) and $m = m_0$. When its velocity is v, its mass becomes m. Therefore, integrating equation (6),

$$E_k = \int_o^{E_k} dE_k = c^2\int_{m_o}^{m} dm = c^2(m - m_0)$$

$\therefore$ $$E_k = mc^2 - m_oc^2 \quad ...(7)$$

This is the relativistic formula for K.E.

When the body is at rest, the internal energy stored in the body is m_0c^2. m_0c^2 is called the rest mass energy. The total energy (E) of the body is the sum of K.E. (E_k) and rest mass energy (m_0c^2).

$\therefore$ $$E = E_k + m_0c^2 = (mc^2 - m_0c^2) + m_0c^2 = mc^2.$$

$\therefore$ $$E = mc^2$$

This is Einstein's mass-energy relation.

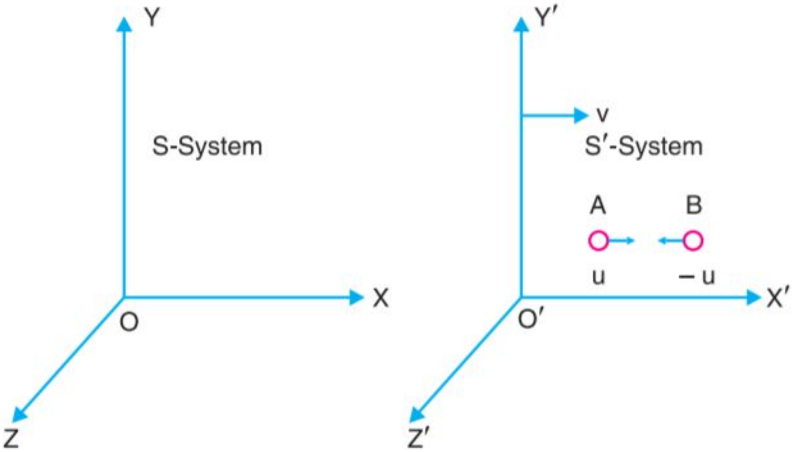
Variation of Mass with Velocity

Derivation. Consider two systems S and S'. S' is moving with a constant velocity v relative to the system S, in the positive X-direction (Fig. 1.8). Suppose that in the system S', two exactly similar elastic balls A and B approach each other at equal speeds (*i.e.*, u and $-u$). Let the mass of each ball be m in S'. They collide with each other and after collision coalesce into one body. According to the law of conservation of momentum,

Momentum of ball A + momentum of ball B = momentum of coalesced mass.

Or $mu + (-mu)$ = momentum of coalesced mass = 0.

Thus the coalesced mass must be at rest in S' system.



Let us now consider the collision with reference to the system S.

Let u_1 and u_2 be the velocities of the balls relative to S. Then,

$$u_1 = \frac{u+v}{1+uv/c^2} \qquad ...(1)$$

and

$$u_2 = \frac{-u+v}{1-uv/c^2} \qquad ...(2)$$

After collision, velocity of the coalesced mass is v relative to the system S.

Let mass of the ball A travelling with velocity u_1 be m_1 and that of B with velocity u_2 be m_2 in the system S. Total momentum of the balls is conserved. Therefore,

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)\, v \qquad ...(3)$$

Substituting for u_1 and u_2 from equations (1) and (2), we have,

$$m_1\left[\frac{u+v}{1+uv/c^2}\right]+m_2\left[\frac{-u+v}{1-uv/c^2}\right]=(m_1+m_2)v$$

or $$m_1\left[\frac{u+v}{1+\frac{uv}{c^2}}-v\right] = m_2\left[v-\frac{-u+v}{1-\frac{uv}{c^2}}\right]$$

or $$m_1\left[\frac{u+v-v-uv^2/c^2}{1+uv/c^2}\right] = m_2\left[\frac{v-uv^2/c^2+u-v}{1-uv/c^2}\right]$$

or $$m_1\left[\frac{u\left(1-\frac{v^2}{c^2}\right)}{1+\frac{uv}{c^2}}\right] = m_2\left[\frac{u\left(1-\frac{v^2}{c^2}\right)}{1-\frac{uv}{c^2}}\right]$$

or $$\frac{m_1}{m_2} = \frac{1+uv/c^2}{1-uv/c^2} \qquad ...(4)$$

Also, $$1-\frac{u_1{}^2}{c^2} = 1-\frac{\left\{\frac{u+v}{c}\right\}^2}{\left(1+\frac{uv}{c^2}\right)^2}$$

$$\frac{1+\frac{u^2v^2}{c^4}+\frac{2uv}{c^2}-\frac{u^2}{c^2}-\frac{v^2}{c^2}-\frac{2uv}{c^2}}{\left(1+\frac{uv}{c^2}\right)^2}$$

$$= \frac{\left(1-\frac{u^2}{c^2}\right)-\frac{v^2}{c^2}\left(1-\frac{u^2}{c^2}\right)}{\left(1+\frac{uv}{c^2}\right)^2}$$

$$1-\frac{u_1{}^2}{c^2} = \frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{uv}{c^2}\right)^2} \qquad ...(5)$$

Similarly, $$1-\frac{u_2^2}{c^2} = \frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{uv}{c^2}\right)^2} \qquad ...(6)$$

Dividing equation (6) by equation (5),

$$\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}} = \frac{\left(1+\frac{uv}{c^2}\right)^2}{\left(1-\frac{uv}{c^2}\right)^2}$$

or
$$\frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}} = \frac{1+\frac{uv}{c^2}}{1-\frac{uv}{c^2}} \qquad ...(7)$$

From equations (7) and (4), $\dfrac{m_1}{m_2} = \dfrac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}$

or
$$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} \qquad ...(8)$$

Since the L.H.S. and R.H.S. of equation (8) are independent of one another, the above result can be true only if each is a constant. Therefore,

$$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} = m_0.$$

The constant denoted by m_0 is called the *rest mass* of the body and corresponds to *zero velocity.*

Thus,
$$m_1 = \frac{m_0}{\sqrt{1-u_1^2/c^2}}$$

In general, if m denotes the mass of a body when it is moving with a velocity v, t n,

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}} \qquad ...(9)$$

This is the relativistic formula for the variation of mass with velocity.

If we put $v \rightarrow c$ in equation (9), we have $m \rightarrow \infty$ *i.e.*, an object travelling at the speed of light would have infinite mass. Thus, Eqn. (9) shows that no material body can have a velocity equal to, or greater than the velocity of light.