MCA Part III

Paper- XXI

Topic: Fuzzy Logic

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INTRODUCTION

In our day to day life we have problems, which cannot be given discrete values like true or false. For example, consider the statement: If the pizza is **very hot**, keep it in open for some time to get normal to eat.

In the above statement, for a number of words/phrases including '**very hot'** it is not possible to tell when exactly pizza is very hot, when pizza is at normal temperature, when exactly pizza is comfortable for eating. These temperature consideration may vary from person to person. Some may like to have very hot pizzas and some when the pizza is cold.

Some other cases of Fuzziness in a Natural Language

Healthy Person: we cannot define health easily. It is difficult to enumerate all the parameters that **determine** health. Further, it is even more difficult to tell for what value of a particular parameter, one is healthy or otherwise.

Fat or thin person: It is not possible to tell exactly up to exactly what size, one is to be called **fat** because fattening is a **continuous** process.

Sweetening of Milk: if we add small sugar cube one at a time to glass of milk, and go on adding up to, say, 10 small cubes. Initially, without sugar, the milk may not be sweet. However, with addition of each one small sugar particle cube, the sweetness **gradually** increases. It is not possible to say that after addition of 10 small cubes of sugar, the milk becomes sweet, and, till addition of 9 small cubes, it was not sweet. Sweetness may vary from person to person. Thus an accurate value cannot be assigned to it. Thus when words like good, bad, healthy, beautiful, hot, cold etc., are used in natural languages it becomes difficult to give a complete value like true or false to it. Each of these words does not denote something constant, but is a sort of linguistic variable. The context of a particular usage of such a word may delimit the scope of the word as a linguistic variable. The range of values, in some cases, for some phrases or words, may be very large as can be seen through the following three statements:

- Mughals ruled India for a very long period (about 100 of years)
- It has been hot since very long period (say about eight months).
- I had to wait for the tickets in the queue for a long period (about three hours).

In Fuzzy theory we can handle such situations. A **Fuzzy theory** may be thought as a technique of providing '**continuation**' to the otherwise binary disciplines like Set Theory, PL and FOPL.

Let us consider an example of our being comfortable with imprecision than precision. The statement: 'The sky is covered with black clouds' is more comprehensible to human beings than the statement: 'The cloud cover of the sky is 90 %'.

This is because of the fact that, we, the human beings are still much better than computers in **qualitative** reasoning. Because of better qualitative reasoning capabilities just by looking at the eyes only and/or nose only, we may recognize a person.

- Just by taking and feeling a small number of grains from cooking rice bowl, we can **tell** whether the rice is properly cooked or not.
- Just by looking at few buildings, we can identify a locality or a city.

Equating Computers to Human Beings

To achieve human capability in solving the problems discussed above, computers must be able to solve problems for which only incomplete and/or imprecise information/knowledge is available.

Preparing the Solutions

As we know that for any problem, the plan of the proposed solution and the relevant information is put in the computer in a form which is acceptable to the computer.

We also know that the problems to be solved with the help of computers are, in the first place, for the human beings and then, the complete plan of the solution is also designed and prepared by human beings. Therefore, we can conclude that the human beings models the problem as well as their solutions to be fed in the computer.

Thus it can be said that the computer is mainly used for execution, because computers fast and can repeat the same activity many times without getting bored.

Thus we can say that we as the human beings have the capability of sensing problems, have desires for solving the problems and express the problems and the plan of a solution using general words of a natural language. We use computers to solve the problems, because of their speed, accurateness and executional power. Computers function better, when the information is given to the them are deterministic that is, when the **steps of solution** are generally defined and clear, involving no duplicity and in terms of **mathematical entities** like numbers, sets, functions, relations, matrices, vectors, graphs, matrices, trees, records, etc.

In order to meet requirements of mutual conflict we need to use natural language in which the human beings are comfortable. Human being can think a problem in these languages properly and thus plan the appropriate solution for it. Use precision of a formal system, with which computers will operate efficiently and execute the solution, as planned by human beings using a new formal system. A new formal system is, Fuzzy system which is based on the concept of 'Fuzzy'. Fuzzy was the concept suggested by L. Zadeh for the first time in 1965.

To understand Fuzzy systems, we need to define two statements first that is the difference between a precise (exact) statement and an imprecise (lacking exactness) statement.

An example of precise statement is: **'If you secure more than 75% of marks then you will be awarded a distinction'.**

An example of **imprecise** statement is: **'If the humidity increases to 85% today then there is the probability of rain'.**

The 'Fuzzy' concept when applied as a prefix/adjective to mathematical entities like set, relation, functions, tree, etc., helps us in modelling the imprecise data, information or knowledge through mathematical tools.

Difference between **Crisp Set/Relation and Fuzzy Set/Relation:** Normal sets are basically called the crisp set.

The fuzzy sets are defined as mathematical entities that **capture imprecise concepts. A**n example of the concept: tall. In Indian context, we may say, a **male adult**, is tall.

In **modelling the vague concept like 'tall', through fuzzy sets,** the numbers in the **closed set [0, 1] of reals** may be used as follows:

'**Definitely tall**' may be represented as '**tallness having value 1**'

'**Not at all tall**' may be represented as '**Tallness having value 0'**

'**A little bit tall**' may be represented as '**tallness having value say .2'**.

'**Slightly tall**' may be represented as '**tallness having value say .4'**.

'**Reasonably tall**' may be represented as '**tallness having value say .7'**.

Similarly, the values of other concepts or, rather, other **linguistic variables like happy, bad, beautiful**, etc. may be considered **in terms of real numbers between 0 and 1.**

Let us think of five male persons of an organisation, like, Ram, Shyam, Hari, Aslam, John, with heights 5' 3", 6' 2", 5' 8", 4' 7", 5' 6" respectively.

Then had we talked only of crisp set of tall persons, we would have denoted the

But, a fuzzy set, representing tall persons, include **all the persons along with respective degrees of tallness**. Thus, **in terms of fuzzy sets,** we write:

Set of tall persons in the organisation = {Shyam}

Tall = {Ram/.2; Shyam/1; Hari/.7; Aslam/0; John/.4}.

The values .2, 1, .7, 0, .4 are called **membership values or degrees**:

Note: The elements which have value 0 may be dropped for example:

Tall may also be written as Tall = {Ram/.2; Shyam/1; Hari/.7; John/.4}, dropping Aslam, who is associated with the degree zero.

If we **define short** as exactly **opposite of Tall** we may say:

Short = {Ram/.8; Shyam/0; Hari/.3; Aslam/1; John/.6}

Understanding the difference between Crisps set and Fuzzy sets:

Crisp sets: In crisps sets , we have the concepts of Equality of sets, subset of a set, and member of a set, as shown below by the following examples:

(i) **Equality of two sets**

Let A = {1, 2, 3, 5}

B = {2, 1, 3, 5}

C = {1, 2, 4, 5}

be three given sets.

Then, Set A is equal to set B denoted by A = B. But A is not equal to C, denoted by A ≠ C.

(ii) **Subset of a Set**

Consider sets: A = {1, 2, 3, 4, 5, 6, 7}

B = {1, 3, 5}

C = {4, 9}

Then B is a subset of A, denoted by B ⊂ A. Also C is not a subset of A, denoted by C ⊄ A.

(iii) Belongs to/ is a member of If A={1,3,4,5}

Then each of 1, 3, 4 and 5 is called an element or member of A and the fact that 1 is a member of A is denoted by **1 ∈ A.**

Concepts in Fuzzy Sets

To define for fuzzy sets, let us illustrate the concepts through an example:

Let X be the set on which fuzzy sets are to be defined, e.g.,

X = {Ram, Shyam, Hari, Aslam, John}. Then X is called the **Universal Set.**

(i) Degree of Membership: In respect of fuzzy sets, we do not speak of just 'membership', but speak of 'degree of membership'.

In the set

A = {Ram/.2; Shyam/1; Hari/.7; John/.4}, Degree (Ram) = .2,

degree (Hari) =.4

(ii)Equality of Fuzzy sets: Let A, B and C be fuzzy sets defined on X as follows:

Let A = {Ram/.2; Shyam/1; Hari/.7; John/.4} B = {John/.4, Ram/.2; Shyam/1; Hari/.7}.

Then, as degrees of each element in the two sets, equal; we say fuzzy set A equals fuzzy set B, denoted as A = B

However, if C = {John/.2, Ram/.4; Shyam/1; Hari/.7}, then A ≠ C.

(iii) Subset/Superset

We know that,

The **Set of 'Very Fat' people** should be a **subset** of the set of **Fat people**.
If the **degree of 'Fatness'** of a person **is say .5** then degree of **'Very Fat'** for the person should be **lesser say .3**.

Combining the above two ideas we, may say that if
(i)

A = {Ram/.2; Shyam/1; Hari/.7; John/.4} and

B = {Ram/.2, Shyam/.9, Hari/.6, John/.4}and further,

C = {Ram/.3, Shyam/.9, Hari/.5, John/.4},

Then, in view of the fact that for each element, degree in A is greater than or equal to degree in B, **B is a subset of A** denoted as B ⊂ A. However, degree (Ram) = .3 in C and degree (Ram) =.2 in A, therefore, C is **not** a subset of A. On the other hand degree (Hari) = .5 in C and degree (Hari) = .7 in A, therefore, A is also not a subset of C. We generalize the ideas illustrated through examples above:

Let **A and B be fuzzy sets** on the universal set

$X = \{x_1, x_2, \ldots, x_n\}$
(X is called the Universe or Universal set)

Such that,

A = {x1/r1, x2/r2, …., xn/rn}
B = {x1/s1, x2/s2, …., xn/sn}

with that $0 \le r_i , s_i \le 1$.
(ii)
Then fuzzy set A equals fuzzy set B, denoted as A = B, if and only if $r_i = s_i$ for all i = 1,2,….,n.

Further if **and** $w \le v_i$ for **all i. then B is a fuzzy subset of A.**

Example: Let X = {Ram, Shyam, Hari, Aslam, John} A = {Ram/.2; Shyam/1; Hari/.7; John/.4} B = {Ram/.2, Shyam/.9, Hari/.6, John/.4}

Then B is a fuzzy subset of A.
In respect of fuzzy sets vis-à-vis (crisp) sets, we may note that:

Corresponding to the concept of '**belongs to**' of **(Crisp) set**, we use the concept of '**degree of membership**' for fuzzy sets.

It may be noted that every **crisp set** may be thought of as a **Fuzzy Set,** but **not conversely**. For example, if **Universal set is**

(iii)

X = {Ram, Shyam, Hari, Aslam, John} and

A = set of those members of X who are **at least graduates,** say,
= {Ram, Hari, Aslam}, then we **can rewrite A as a fuzzy set** as follows:

A = {Ram/1; **Shyam**/0; Hari/1; Aslam/1; **John**/0}, in which degree of each member of the crisp set, is taken as one and degree of each element of the universal set which does not appear in the set A, is taken as zero.

However, conversely, a fuzzy set **may not** be written as a **crisp set.** Let C be a fuzzy set denoting **Educated People,** where **degree of education is defined as** follows:

Degree of education (Ph.D.) = 1

Degree of education (MA/MSc.) = 0.85

Degree of education (BA/BSc.) = .6

Degree of education (10 + 2 level) = 0.4

Degree of education (8^{th} Standard) = 0.1

Degree of education (less than 8th) = 0.

Let us C = {Ram/.85; Shyam/.4; Hari/.6; Aslam/1; John/0}.

Then, we cannot think of C as a crisp set.

Some more concepts in of fuzzy sets.

(i)Support set of a Fuzzy Set: A support of a fuzzy set within a Universal set X is the crisps set that contains all the elements of X that have non-zero membership grades in support of a fuzzy set a. For example:

A = {Ram/.85; Shyam/.4; Hari/.6; Aslam/1; John/0}.

Therefore, support set D of A can be written as:

Support of **A = D = {Ram, Shyam, Hari, Aslam},** where **the element John does not belong to D, because, it has degree 0 in A.**

(ii) Fuzzy Singleton is a fuzzy set in which there is exactly one element which has positive membership value.

Example:

Let us define a fuzzy set YOUNG on universal set X in which degree of YOUNG is zero if a person in X is above 50 years and Degree of Young is .2 if a person is between 45 and 50 years and so on:

YOUNG = A = {Ram/0; Shyam/0; Hari/.2; Aslam/0; John/0}, then support of YOUNG= **{Hari}** and hence old is a fuzzy singleton.

FUZZY OPERATORS AND ARITHMETIC

For Crisp sets, we have the operations like **Union, intersection and complementation**, as given below by the example:

Let $X = \{x_1, x_2, \ldots, x_{10}\}$

$A = \{x_2, x_3, x_4, x_5\}$
$B = \{x_1, x_3, x_5, x_7, x_9\}$

Then $A \cup B = \{x_1, x_2, x_3, x_4, x_5, x_7, x_9\}$

$A \cap B = \{x_3, x_5\}$

A' or $X \sim A = \{x_1, x_6, x_7, x_8, x_9, x_{10}\}$

The concepts of Union, intersection and complementation for **crisp sets may be extended to FUZZY** sets after observing that for crisp sets A and B, we have:

(i) A ∪ B is the **smallest** subset of X **containing** both A and B.

(ii) A ∩ B is the **largest** subset of X **contained in** both A and B.

(iii) **The complement A' is such that:**

(a) A and A' **do not have any element in common** and

(b) Every element of the universal set **is in either A or A'.**

Union, Intersection, Complementation of Fuzzy Sets:

Let us recall that

(i) when a crisps set is treated as fuzzy set then membership in a crisp set is indicated by degree/value of membership as 1 (one) in the corresponding Fuzzy set, non-membership of a crisp set is indicated by degree/value of membership as **zero** in the corresponding Fuzzy Set.

Thus, **smaller the value** of degree of membership, a sort of **lesser it is a member** of the Fuzzy set.

(ii) While taking union of Crisp sets, members of both sets are included, and none else. However, in each Fuzzy set, all members of the universal set occur but their degrees determine the level of membership in the fuzzy set.

The facts under (i) and (ii) above, helps us to define:

(i) **Union of two fuzzy sets** A and B, is the set C with the same universe as that of A and B such that, the degree of an element of C is equal to the **MAXIMUM** of degrees of the element, in the two fuzzy sets.

(if Universe A ≠ Universe B, then take Universe C as the union of the universe A and universe B)

(ii) **Intersection C of two fuzzy sets A and B is the fuzzy set in which**, the degree of an element of C is equal to the **MINIMUM** of degrees in the two fuzzy sets.

Example:

A = {Ram/.85; Shyam/.4; Hari/.6; Aslam/1; John/0}

B = {Ram/.75; Shyam/.6; Hari/0; Aslam/.8; John/.3}

Then,

A ∪ B = {Ram/.85; Shyam/.6; Hari/.6; Aslam/1; John /.3}

A ∩ B = {Ram/.75; Shyam/.4; Hari/0; Aslam/.8; John/0}

and, the complement of A denoted by A′ is given by

C′ = {Ram/.15; Shyam/.6; Hari/.4; Aslam/0; John /1}

Properties of Union, Intersection and Complement of Fuzzy Sets:

The following properties which hold for ordinary sets, also, hold for fuzzy sets

(a) Commutativity

(i) A ∪ B = B ∪ A

(ii) A ∩ B = B ∩ A

(b) Associativity

(i) (A ∪ B) ∪ C = A ∪ (B ∪ C)

(ii) (A ∩ B) ∩ C = A ∩ (B ∩ C)

(c) Distributivity

(i) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) DeMorgan's Laws

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

(e) Involution or Double Complement

$(A')' = A$

(f) Idempotence

$A \cap A = A$

$A \cup A = A$

(g) Identity

$A \cup U = U \quad A \cup U = A$

$A \cap \varphi = A \quad \varphi \cap A = \varphi$

Where,

φ: empty fuzzy set = {x/0 with x ∈ U} and

U: universe = {x/1 with x∈ U}

Operations that are Unique to Fuzzy Sets

Operations like **concentration, dilation, height, normalization and α-cut** are unique to Fuzzy sets and cannot be discussed for (crisp) sets.

(1) Concentration of a set A is defined as

$$\mathbf{CON\ (A) = \{x/m^2_A(x) |\ x \in U\}}$$

Example:

If A = {Ram/.5; Shyam/.9; Hari/.7; Aslam/0; John/.2}

Then,

CON (A) = {Ram/.25; Shyam/.81; Hari/.49; Aslam/0; John/.04}.

We can see that in concentration values being between 0 and 1 become smaller on squaring, in other words the value concentrates towards 0.

(2) **Dilation (Opposite of Concentration)** of a fuzzy set A is defined as

DIL (A) = {x/ mA (x)|x ∈U}

Example:

If A = {Ram/.5; Shyam/.9; Hari/.7; Aslam/0; John/.2}

then

DIL (A) = {Ram/.7; Shyam/.95; Hari/.84; Aslam/0; John/.45}

The associated values that are between 0 and 1, on taking square-root get increased.

(3) **Height of Fuzzy Set:** The height h(A), of a fuzzy set A is the largest membership grade obtained by any element in that set, given by:

$$(A) = \sup_{x \varepsilon A} A(x)$$

A fuzzy set is called Normal when h(A) =1 and called subnormal when h(A)=0.

(4) Normalization of a fuzzy set, **is defined as**

$$\text{NORM}(A) = \{m_A(x)/\max \mid x \varepsilon A\}$$

NORM (A) and is a fuzzy set in which membership values are obtained by dividing values of the membership function of A **by the maximum membership function.**

The resulting fuzzy set, called the **normal**, (or **normalized) fuzzy set**, has the maximum of membership function value of 1.

Example:

If A = {Ram/.5, Shyam/.9, Hari/.7, Aslam/0, John/.2}

Norm (A) = {Ram/(.5 ÷.9 = .55.), Shyam/1, Hari /(.7 ÷.9 = .77.), Aslam/0, John/(.2 ÷.9 = .22.)}

Note: If one of the members has value 1, then Norm (A) = A,

MEMBERSHIP FUNCTIONS

There are largely three types of memberships in fuzzy:

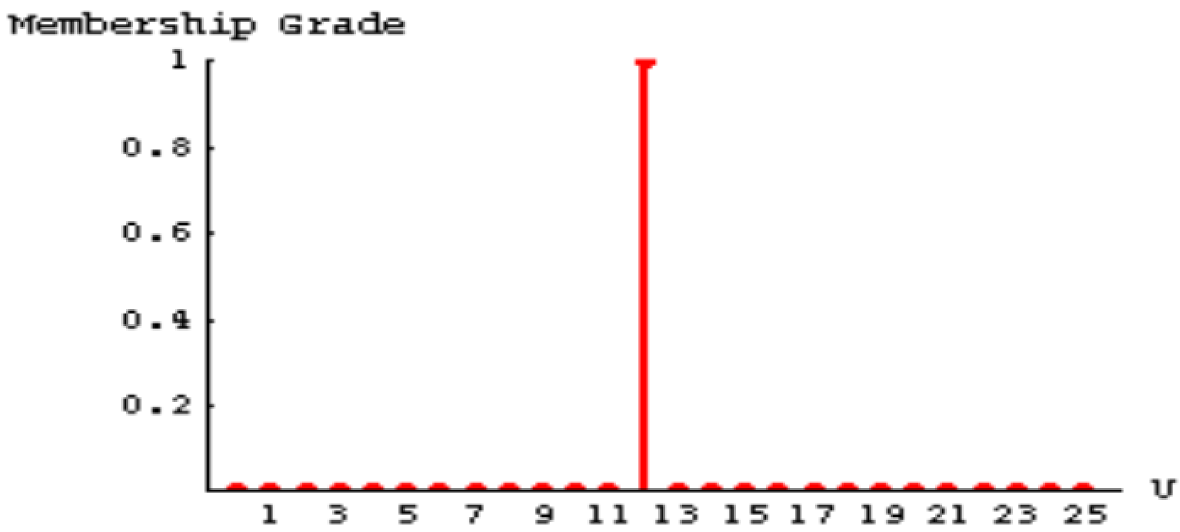
(i) Singleton fuzzifier

The input is converted into fuzzy singletons

A(u) = 1 at u,

A(u) = 0 elsewhere

This simplifies calculations but cannot suppress noise in the input.

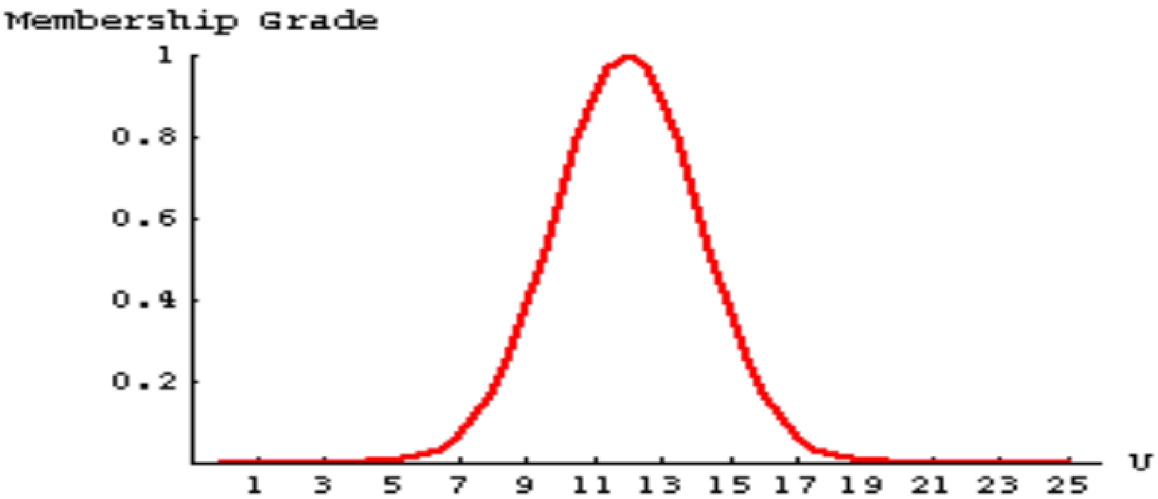


(ii) Gaussian fuzzifier

The input is converted into Gaussian FS

$A(u) = \exp[-(u1-u1^*)/a1]^2 * \ldots * \exp[-(un-un^*)/an]^2$

This simplifies calculations if MFs are Gaussian and also suppress noise in the input.

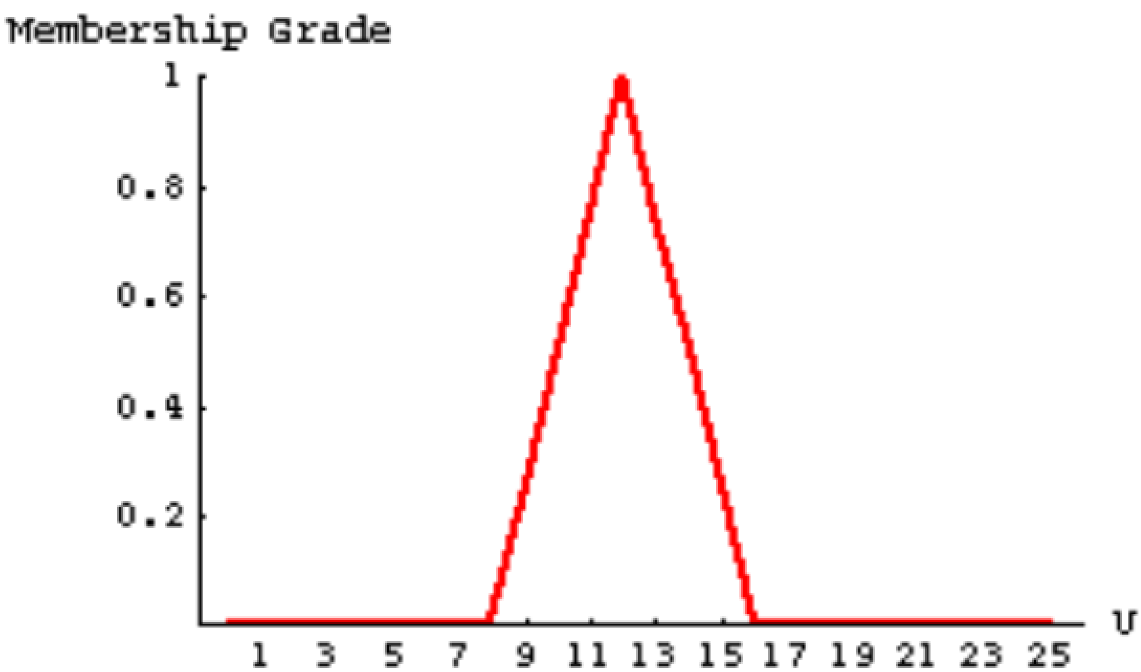


(iii) Triangular or Trapezoidal fuzzifier

The input is converted into triangular fuzzy set

A(u) = (1- |u1-u1*|/b1)*…* (1- |un-un*|/bn)

This simplifies calculations if MFs are triangular and suppress noise in the input.



FUZZY RELATIONS

We that a relation from a set A to a set B is a subset of A x B.

For example, The relation of father may be written as {{Dasrath, Ram), …}, which is a subset of A × B, where A and B are sets of persons living or dead.

The relation of Age may be written as:

{(Ram, 43.7), (Shyam, 25.6),…..}, where A is set of living persons and B is set of numbers denoting years.

In fuzzy sets, every element of the universal set occurs with some degree of membership.

Thus, a fuzzy relation is a fuzzy set defined on the Cartesian product of crisps sets X1, X2, …..Xn, where tuples{x1, x2, …..xn} may have varying degrees of membership within the relation. The membership grade indicates the strength of the relation present between the elements of the tuple. Thus, a **relation from A to B, where we assume A and B as crisp sets, is a fuzzy set, in which with each** element of A × B is associated a degree of membership between zero and one.

A fuzzy relation can also be conveniently represented by an n-dimensional membership array whose entries corresponds to n-tuples in the universal set. These entries take values representing the membership grades of the corresponding n-tuples.

Example 1:

- X = {$x1$, $x2$}, Y = {$y1$, $y2$}, and Z = {$z1$, $z2$, $z3$}
  Consider the following fuzzy relations:

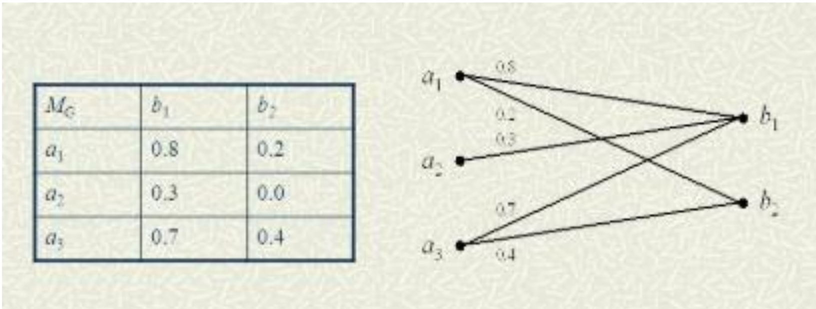
$$\underset{\sim}{R} = \begin{matrix} x_1 \\ x_2 \end{matrix} \overset{\begin{matrix} y_1 & y_2 \end{matrix}}{\begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix}} \quad \text{and} \quad \underset{\sim}{S} = \begin{matrix} y_1 \\ y_2 \end{matrix} \overset{\begin{matrix} z_1 & z_2 & z_3 \end{matrix}}{\begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix}}$$

Then the resulting relation, T, which relates elements of universe X to elements of universe Z,

$$\mu_T(x1, z1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

$$\underset{\sim}{T} = \begin{matrix} x_1 \\ x_2 \end{matrix} \overset{\begin{matrix} z_1 & z_2 & z_3 \end{matrix}}{\begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix}}$$

Example 2:



M_G	b_1	b_2
a_1	0.8	0.2
a_2	0.3	0.0
a_3	0.7	0.4

Fuzzy graph

Reasoning in Fuzzy

Some well-known Crisp Reasoning Operators are:

(i) AND

(ii) OR
(iii) NOT
(iv) IF P THEN Q
(v) P IF AND ONLY IF Q

Corresponding to each of these operators, there is a fuzzy operator discussed and defined below. Let us assume that P and Q are fuzzy propositions with associated degrees, respectively, deg (P) and deg (Q) between 0 and 1.

The **deg (P) = 0 denotes P is False** and

deg (P) =1 denotes P is True.

Then the operators are defined as follows:

(i) Fuzzy AND to be denoted by ∧ , is defined as follows:

For given fuzzy propositions P and Q, the expression P ∧ Q denotes a fuzzy proposition with Deg (P ∧ Q) = **min** (deg (P), deg (Q))

Example 1: Let P: Ram is tall with deg (P) = .6

Q: Ram is educated with deg (Q) = .3
Then **P ∧ Q denotes**: 'Ram is tall and educated' with

degree (**(min)** {.7, .4}) = **.3**

(ii) Fuzzy OR to be denoted by ∨, is defined as follows:

For given fuzzy propositions P and Q, P ∨ Q is a fuzzy proposition with Deg (P ∨ Q) = **max** (deg (P), deg (Q))

Example 2: Let P: Ram is tall with deg (P) = .6

Q: Ram is educated with deg (Q) = .3

Then P ∨ Q denotes: ‘Ram is tall or educated’ with degree (**(max)** {.7, .4}) = **.6**

QUESTIONS

1. Explain the difference between crisps set and fuzzy set.
2. Find some examples of fuzzy variables in your daily life.
3. Describe the concept of fuzzy set in your own words with examples.
4. Discuss different types of operations on fuzzy sets with examples.
5. Discuss α-cut, and height of a fuzzy set with an example.
6. Discuss equality and subset relationship for the following fuzzy sets defined on the Universal set X = { a, b , c, d, e}

   A = {a/.3, b/.6, c/.4 d/0, e/.7}

   B = {a/.4, b/.8, c/.9, d/.4, e/.7}

   C = {a/.3, b/.7, c/.3, d/.2, e/.6}

7. For the given fuzzy sets below:

   X = {a/.5, b/.6, c/.3, d/0, e/.9} and

   Y = {a/.3, b/.7, c/.6, d/.3, e/.6},

   Find the fuzzy sets **X ∩ Y, X ∪ Y and (X ∩ Y)'**