E-content- B.Sc. (Mathematics), Degree-1

Subject- Mathematics, Paper-II

Topic-3 Analytical Geometry (3-D)

Name: Dr. Santosh Kumar (CSIR JRF, SRF, Ph.D.)

Guest Faculty, Department of Mathematics,

Patna Science College, Patna-800005

Mobile No. 8210642534

E-Mail-santoshrathore.kumar20@gmail.com

CONE AND RIGHT CIRCULAR CONE

Definition of Cone:-

A cone is a surface generated by a variable straight line passing through a fixed point and satisfying one more condition i.e. intersecting a given curve or touching a given surface.

The fixed point is called the vertex and the given curve (or surface) is called the guiding curve (or guiding surface) of the cone. The variable straight line is known as the generator of the cone.

A cone whose equation is of second degree is known as quartic cone or quadratic cone.

Cone with vertex at the origin. To prove that the equation of the cone with vertex at the origin is a homogeneous second degree equation in *x*,*y* and *z*.

Let the general equation of the second degree in *x*, *y*, *z* viz.

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$
...(i)

represent a cone with vertex at the origin *O*.

Let $P(x_1,y_1,z_1)$ be any point on the cone. The equation of the generator OP is $\frac{x-0}{x_1}=\frac{y-0}{y_1}=\frac{z-0}{z_1}$
...(ii)

Any point *Q* on the this generator is (rx_1,ry_1,rz_1). As *OP* is a generator of the cone (i), so every point on it like *Q* must lie on cone (i) for all values of *r*, which means that

$$r^2\left(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1\right)+2r\left(ux_1+vy_1+wz_1\right)+d=0,$$

must be an identity and the conditions for the same are

$$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0 \qquad \text{...(iii)}$$

$$ux_1+vy_1+wz_1=0 \quad \text{...(iv)} \qquad \text{and} \quad d=0 \qquad \text{...(v)}$$

From (iv) we conclude that $P(x_1, y_1, z_1)$ is a point on a plane $ux + vy + wz = 0$ if u, v and w are not all zero and this is against hypothesis. So u, v and w must be all zero. Also (v) is obvious as the cone passes through the origin.

Hence from (i) the equation of the cone with vertex at the origin is
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$ ...(vi)

which is a homogeneous equation of second degree in x, y and z.

Conversely, every homogeneous equation of second degree in x,y and z represents a cone with its vertex at the origin.

It is obvions that if $P(x_1, y_1, z_1)$ satisfies the equation (vi) then for all values of r, the point (ry_1, ry_1, rz_1) also satisfies (vi) *i.e.* all points on the generator OP lie on the surface of the cone *i.e.* the line OP lies entirely on the cone.

Thus the surface is generated by straight lines through the origin and hence it is a cone with its vertex at the origin.

COROLLARY. If the line $x/l = y/m = z/n$ is a generator of the cone given by $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$ ...(i)

then its direction cosines viz. l, m, n satisfy the equation of the cone.

Any point on the given generator is (lr, mr, nr). If this lies on the cone (i) then we have $al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$ ...(ii)

Conversely if the relation (ii) holds good *i.e.* if the direction ratios of a straight line which always passes through a fixed point satisfy a homogeneous equation, then this line is a generator of a cone whose vertex is at the fixed point.

General equation of a cone of second degree which passes through the co-ordinate axes.

If the cone passes through the co-ordinate axes, then its vertex must be at the origin and as such its equation must be of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad ...(i)$$

Also the d.c.' s of the co-ordinate axes are 1, 0, 0; 0, 1, 0 and 0, 0, 1 which should satisfy (i).

So we get $a=0, b=0$ and $c=0$. Hence the required equation of the cone through the axes is $fyz+gzx+hxy=0$.

Problem (1):- Prove that equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=$ represents a cone $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$

Solution:- The given equation is

$$ax^2+by^2+cz^2+2ux+2vy+2wz+d=0 \qquad —(1)$$

Making it homogeneous in $x, y, z, t,$ where t =1,

we get

$$F(x,y,z,t) \equiv ax^2+by^2+cz^2+\alpha uxt+2vyt+2wzt+d=c$$

The given eq^n. will represent a cone it

$$\frac{\partial F}{\partial x}=0, \quad \frac{\partial F}{\partial y}=0, \quad \frac{\partial F}{\partial z}=0, \quad \frac{\partial F}{\partial t}=0, \text{ where t=1}$$

These gives

$$2ax+2ut=0 \text{ or } x=-\frac{u}{\alpha}, \text{ as t = 1}$$

$$2by+2vt=0 \text{ or } y=\frac{-v}{a}, \text{ as t =1}$$

$$2cz+2wt=0 \text{ or } z=\frac{-w}{c}, \text{ as t = 1}$$

and

$$2ux+2vy+2wz+2dt=0$$
$$ux+vy+wz+d=0 \text{ as t =1} \qquad —(2)$$

Substituting the values of x, y, z in (2), we get

$$\frac{u\times -u}{a}+\frac{v\times -v}{a}+\frac{wx-w}{c}+d=0$$

$$\frac{-u^2}{a}+\frac{-v^2}{b}-\frac{w^2}{c}+d=0$$

$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$$

This is the required condition.

Problem (2):- Find the equation of the cone whose vertex is the point (α,β,v) and whose generators intersect the conic $ax^2+2hxy+by^2+\alpha gx+2+y+c=0, z=0$

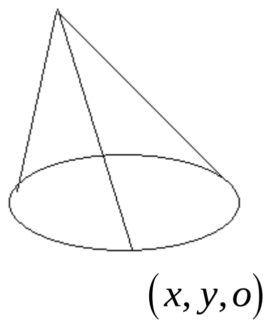
Solution:- Let the eq^n. of a line through the vertex $v(\alpha,\beta,\gamma)$ of the cone be

$$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{—(1)}$$

If it meets the plane z =o dt the pt. P then from (1)

$$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{o-\gamma}{n}.$$

or $x=\frac{\alpha-\ell}{n}\gamma, y=\frac{\beta-m}{n}\gamma$



The point $\left(\frac{\alpha-\ell}{n},\frac{\beta-m}{n}\gamma,0\right)$

Lie on the guiding curue.

$$ax^2+2hxy+by^2+2gx+2ty+c=o, z=0$$

If $a\left(\alpha-\frac{\ell}{n}\gamma\right)^2+2h\left(\alpha-\frac{\ell}{n}\gamma\right)\left(\beta-\frac{m}{n}\gamma\right)+b\left(\beta-\frac{m}{n}\gamma\right)^2$

$$+2g\left(\alpha-\frac{\ell}{n}\gamma\right)+2+\left(\beta-\frac{m}{n}\gamma\right)+c=0 \qquad \text{—(2)}$$

Eliminating l,m,n from (1) & (2), we get the eq^n. of the cone generated by the

line (1) $\frac{\ell}{n}=\frac{x-\alpha}{z-\gamma}$ and $\frac{m}{n}=\frac{y-\beta}{z-\gamma}$

$$\therefore \quad \alpha-\frac{\ell}{n}\gamma=\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma=\frac{z\alpha-x\gamma}{z-\gamma}$$

and $\beta-\frac{m}{n}\gamma=\beta-\left(\frac{y-\beta}{z-\gamma}\right)^{\gamma}=\frac{\beta z-\gamma y}{z-\gamma}$

Substituting these values in (2), we get the required eq^n. of the cone is

$$\alpha\left(\frac{\alpha-x\gamma}{z-\gamma}\right)^2+2n\left(\frac{z\alpha-x\gamma}{z-\gamma}\right)\left(\frac{\beta z-\gamma y}{z-\gamma}\right)+b\left(\frac{\beta z-\gamma y}{z-\gamma}\right)2+g\left(\frac{z\alpha-x\gamma}{z-\gamma}\right)+2+\left(\frac{\beta z-\gamma y}{z-\gamma}\right)+C=O$$

$$\Rightarrow\ \alpha(\alpha z-x\gamma)^2+2n(z\alpha+x\gamma)(\beta z-\gamma y)+b(\beta z-\gamma y)+2g$$
$$(z\alpha-x\gamma)(z-\gamma)+2+(\beta z-\gamma y)(z-\gamma)+C(z-v)2=O$$

Problems (3):- Find the condition that the general homogeneous equation of the 2nd degree should represent a right circular cone.

Solution:- The general homogeneous equation of the second degree is

$$\alpha x^2+by^2+cz^2+2yz+2g2x+2hxy=O \qquad —(1)$$

The equation of the right circular cone whose vertex is the origin is

$$(\ell x+my+n_z)^2=(\ell^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta.$$

Where θ is the semi-vertical angler of the cone.

Writing $(\ell^2+m^2+n^2)\cos^2\theta=\lambda,$ the equation of the right circular cone becomes

$$(\ell^2+my+nz)^2=\lambda(x^2+y^2+z^2)$$

$$\Rightarrow \ell^2x^2+m^2y^2+n^2z^2+2\ell mxy+2mnyz+2\ell nxz=\lambda x^2+\lambda y^2+\lambda z^2$$

$$\Rightarrow \ell^2x^2-\lambda x^2+m^2y^2-\lambda y^2+n^2z^2-\lambda z^2+2\ell nzx+2\ell mxy+2mzyz=O$$

$$\Rightarrow x^2(\ell^2-\lambda)+y^2(m^2-\lambda)+z^2(n^2-\lambda)+2mnyz+2n\ell zx+2\ell xy=O \qquad —(2)$$

Compeering. (1) & (2) we get

$$\frac{a}{a^2-\lambda}=\frac{b}{m^2-\lambda}=\frac{c}{n^2-\lambda}=\frac{f}{mn}=\frac{g}{n\ell}=\frac{h}{\ell m}=k \text{ (say)}$$

$\therefore\ g=nlk, h=lmk$ and $f=mnk$

$$gh=\ell^2mnk^2$$

$$\frac{gh}{f}=\frac{\ell^2nk^2}{mnk}=\ell^2k \qquad —(3)$$

But from the first equality

$$a = \left(\ell^2 - \lambda\right) K$$

$$i.e. \quad \ell^2 - \lambda = \frac{9}{k} \qquad \text{— (4)}$$

Eliminating ℓ between (3) & (4) we get

$$\frac{gh}{fk} - \lambda = \frac{9}{k}$$

$$\Rightarrow \frac{gh}{fk} - \frac{a}{k} = \lambda$$

$$\Rightarrow \frac{gh - af}{f} = \lambda k$$

Similarly,

$$\frac{hf - bg}{g} = \lambda k \text{ and } \frac{fg - ch}{h} = \lambda k.$$

Thus, the required condition is

$$\frac{gh - af}{f} = \frac{hf - bg}{g} = \frac{fg - ch}{h}$$

Problem (4):- Find the equations to the line in which the plane $2x + y - z = o$ cuts the cone $4x^2 y^2 + 3z^2 = 0$.

Solution:- Since the vertex of the cone is origin and the plane passes through origin.

Hence, let the eq^n. of the line be

$$\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n} \qquad \text{— (1)}$$

In which the plane $2x + y - z = o$ wets the cone $4x^2 y^2 + 3z^2 = o$

$$\therefore \; 4\ell^2 m^2 + 3n^2 = o \qquad \text{— (2)}$$

and $\quad 2\ell + m - n = o \qquad$ — (3)

Eliminating ‘n’ between (2) & (3), we get

$$4\ell^2 - m^2 + 3(2\ell + m)^2 = o$$

$$(2\ell - m)(2\ell + m) + 3(2\ell + m)^2 = o$$

$$\Rightarrow \ (2\ell + m)[2\ell - m + 3(2\ell + m)] = O$$

$$\Rightarrow \ (2\ell + m)(4\ell + m) = O$$

$$\Rightarrow \ 2\ell + m = O \Rightarrow \frac{\ell}{m} = \frac{-1}{\alpha}$$

and $\quad 4\ell + m = o \ \Rightarrow \ \frac{\ell}{m} = \frac{1}{4}$

when $\frac{\ell}{m} = \frac{-1}{2}$ then (3) $\Rightarrow n = o$

$$\therefore \ \frac{\ell}{-1} = \frac{m}{2} = \frac{n}{O}$$

When $4\ell + m = o$ then from (3), we have

$$2\ell - n = O$$

or $\quad \frac{\ell}{-1} = \frac{n}{2}$

$$\therefore \ \frac{\ell}{-1} = \frac{m}{4} = \frac{n}{\alpha}$$

Hence the lines of section are

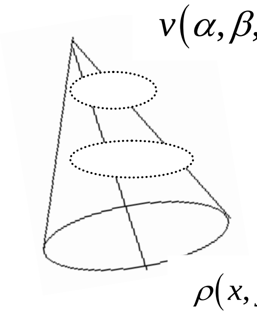
$$\frac{x}{-1} = \frac{y}{\alpha} = \frac{z}{o} \ ; \ \frac{x}{1} = \frac{y}{4} = \frac{z}{\alpha}$$

Problem (5):- Find the equation of the cone whose vertex is (α, β, γ) and the guiding curve is the conec $z = o, ax^2 + by^2 + 2hxy + 2gx + 2fy + czo.$

Solution:- Let the equation of a line through the vertex $V(\alpha, \beta, \gamma)$ of the conc be.

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{— (1)}$$

If it meets the plane z = o at the point P then from (1)



$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

or $\qquad x = \frac{\alpha - \ell\gamma}{n}, y = \beta - \frac{m}{n}\gamma$

The point $\left(\alpha-\frac{\ell}{n}\gamma,\beta\frac{-m\gamma}{n},\text{o}\right)$ will lie on the guiding curve

$$ax+2hxy+by^2+2gx+2fy+c=o,z=o$$

It $a\left(\alpha-\frac{\ell}{n}\gamma\right)^2+2h\left(\alpha-\frac{\ell}{n}\gamma\right)\left(\beta\frac{-m}{n}\gamma\right)+b\left(\beta\frac{-m}{n}\gamma\right)^2$

$$+2g\left(\alpha-\frac{\ell}{n}\gamma\right)+2f\left(\beta-\frac{m}{n}\gamma\right)+\text{c}=\text{o} \qquad \text{— (2)}$$

Elimination, ℓ,m,n from (1) & (2), we get the eq^{n}. of the cone generated by the line (1)

from (1)

$$\frac{\ell}{n}=\frac{x-\alpha}{z-\gamma} \text{ and } \frac{m}{n}=\frac{y-\beta}{z-\gamma}$$

$$\therefore \quad \alpha\frac{\ell}{n}\gamma=\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma=\frac{\alpha z-x\gamma}{z-\gamma}$$

$$\text{and } \beta=\frac{m\gamma}{n}=\beta-\left(\frac{y-\beta}{z-\gamma}\right)^{\gamma}$$

$$=\frac{\beta z-\beta\gamma-y\gamma+\beta\gamma}{z-r}$$

$$=\frac{\beta z\text{ - }y\gamma}{z-\gamma}$$

Substituting these values in (2), the required eq^{n}. of the cone is

$$a\left(\frac{az-\gamma x}{z-\gamma}\right)^2+2h\left(\frac{az-\gamma x}{z-\gamma}\right)\left(\frac{\beta z-\gamma y}{z-\gamma}\right)+b\left(\frac{\beta z-\gamma y}{z-\gamma}\right)^2$$

$$+2g\left(\frac{az-\gamma x}{z-\gamma}\right)+2f\left(\frac{\beta z-\gamma y}{z-\gamma}\right)+\text{ c}=\text{o}$$

$$a(az-\gamma x)^2+2h(az-\gamma x)(\beta z-\gamma y)+b(\beta z-\gamma y)^2$$

$$+\,2g(az-\gamma x)(z-\gamma)+2f(\beta z-\gamma y)(z-\gamma)+C(z-\gamma)^2=\text{O}$$

Problem (6):- Find the equation of a right circular cone whose vertex is (3,2,1) axis is the line $\frac{x-3}{4}=\frac{y-2}{1}=\frac{z-1}{3}$ and semi-vertical angle is 30:.

Sol^n. Let ℓ, m, n be the direction ratios of any generator of the conc. The its direction cosines are

$$\frac{\ell}{\sqrt{\ell^2+m^2+n^2}}, \quad \frac{m}{\ell^2+m^2+n^2}, \quad \frac{n}{\sqrt{\ell^2+m^2+n^2}}$$

and proceed as above problem.

****