E-content- B.Sc. (Mathematics), Degree-1

Subject- Mathematics, Paper-II

Topic-1 Analytical Geometry (3-D)

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SPHERE

1.01. Definition. A sphere is the locus of a point which moves such that its distance from a fixed point always remains constant.

The fixed point is called the centre and this constant distance is called the radius of the sphere.

1.02. The equation of a sphere.

(a) When centre and radius are given (Central From):

Let *C (a,b,c)* be the centre and *r* the radius of the sphere. Then if $P(x, y, z)$ be any point of the surface of the sphere we have *CP*= radius of the sphere=*r* i.e. $CP^2 = r^2$ and therefore

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \qquad \text{......(i)}$$

For example if the centre of a sphere is (1, 2, 3) and radius is 4. Then its equation is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 4^2$

Particular Case: Let origin O (0,0,0) be the centre and *r* the radius of the sphere. Let $P(x, y, z)$ be any point on this sphere. The *OP* = radius of the sphere = *r* (given)

$$\Rightarrow \quad OP^2 = r^2$$
$$\Rightarrow \quad (x-0)^2 + (y-0)^2 + (z-0)^2 = r^2$$
$$\Rightarrow \quad x^2 + y^2 + z^2 = r^2 \qquad \text{...(ii)}$$

which is called the standard form of the equation of a sphere.

(b) General Equation of sphere.

The equation (i) above can be expended and written as

$x^2 + y^2 + z^2 - 2ax - 2by - 2cz - (a^2 + b^2 + c^2 - r^2) = 0$ which is of the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad \text{...(iii)}$$

and this is known as the general form of the equation of a sphere.

Centre and radius for general form.

The equation (iii) can be rewritten as

$(x+u)^2+(y+v)^2+(z+w)^2=u^2+v^2+w^2-d.$

or $\quad [x-(-u)]^2+[y-(-v)]^2+[z-(-w)]^2=u^2+v^2+w^2-d.$

Comparing this with equation (ii) above, we find that the centre and radius of the sphere given by (iii) are $(-u,-v,-w)$ and $\sqrt{(u^2+v^2+w^2-d)}$ respectively.

Hence we conclude that for the sphere (iii),

centre is $(-u,-v,-w)$ and radius = $\sqrt{(u^2+v^2+w^2-d)}$.

Method of writing the centre and radius of a sphere.

(i) Write down the coefficients of x^2, y^2 and z^2 as 1, if these are not so, by dividng the equation by the coefficient of x^2. For example if the equation of the sphere is given as $2x^2+2y^2+2z^2-10x-12y+16z+23=0,$ then we should divide each term by the coefficient 2 of x^2 and write the given equation as

$$x^2+y^2+z^2-5x-6y+8z+(23/2)=0.$$

(ii) Then the coordinates of the centre of the sphere are

$=[-\frac{1}{2}(\text{coefficient of } x), -\frac{1}{2}(\text{coeff. of y}), (-\frac{1}{2}(\text{coeff. of z})]$ and the radius of the sphere $=\sqrt{} = [\frac{1}{2}(\text{coeff. of } x)^2 + (\frac{1}{2}(\text{coeff. of y})^2 + (\frac{1}{2}(\text{coeff. of z})^2]$

—(constant term)] (Remember)

 # Conditions for a sphere. From the general form (iii) of the equation of a sphere, we conclude that

(i) The equation of a sphere must be of second degree in x, y and z.

(ii) The coefficient of x^2, y^2 and z^2 must be equal and

(iii) there should not be terms containing the products *xy*, *yz* and *zx* in this equation. **(Remember)**

Note 1. If $u^2+v^2+w^2-d<0,$ then the radius of the sphere (iii) is imaginary whereas the centre is real. Such a sphere is called pseudo-sphere or a virtual sphere.

Note 2. The equation (iii) of the sphere contains four unknown constants u, v, w and *d* and therefore a sphere can be found to satisfy four conditions.

1.03. Equation of a sphere through four given points.

(Four-Point Form)

Let the co-ordinates of the four given points *A*, *B*, *C* and *D* be $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ and (x_4, y_4, z_4) respectively.

Let the equation of the sphere passing through these four points be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad \text{...(i)}$$

If this sphere passes through the given points *A*, *B*, *C* and *D*, we have

$$(x_1^2 + y_1^2 + z_1^2) + 2ux_1 + 2vy_1 + 2wz_1 + d = 0, \qquad \text{...(ii)}$$

$$(x_2^2 + y_2^2 + z_2^2) + 2ux_2 + 2vy_2 + 2wz_2 + d = 0, \qquad \text{...(iii)}$$

$$(x_3^2 + y_3^2 + z_3^2) + 2ux_3 + 2vy_3 + 2wz_3 + d = 0, \qquad \text{...(iv)}$$

$$(x_4^2 + y_4^2 + z_4^2) + 2ux_4 + 2vy_4 + 2wz_4 + d = 0. \qquad \text{...(v)}$$

Eliminating u, v, w and d from (i), (ii), (iii), (iv) and (v), we get

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Note: In numerical problems, evaluation of determinant takes much time and so the values of u, v, w and d should be found from (ii), (iii), (iv) and (v) and then these should be substituted in (i) to get the required equation.

1.04. Equation of a sphere on the line joining two given points as diameter. (Diameter From).

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two given points.

Let $P(x, y, z)$ be any point on the surface of the sphere, drawn on the line joining *A* and *B* as diameter. Then *AP* is perpendicular to *BP*.

Now the direction ratios of the lines *AP* and *BP* are $x - x_1, y - y_1, z - z_1$ and $x - x_2, y - y_2, z - z_2$ respectively.

As *AP* is perpendicular to *BP*, we have

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0, \qquad \text{...(i)}$$

Which is the required equation.

Problem (1) :-

If the tangent plane to the sphere $x^2+y^2+z^2=r^2$ makes intercepts a,b,c on the co-ordinate axes, prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}$

Solution:-

The eq^n. of the sphere is

$$x^2+y^2+z^2=r^2 \qquad \text{—(1)}$$

The eq$^{n.}$ of the tangent plane to the given sphere at the point (x_1, y_1, z_1) is

$$xx_1+yy_1+zz_1=r^2$$

$$\frac{xx_1}{r^2}+\frac{yy_1}{r^2}+\frac{zz_1}{r^2}=1$$

$$\frac{x}{r^2/x_1}+\frac{y}{r^2/y_1}+\frac{z}{r^2/z_1}=1$$

This tangent plane makes an intercepts a,b,c, on the coordinate axes.

$$\frac{x^2}{x_1}=a, \frac{r^2}{y_1}=b, \quad \frac{r^2}{z_1}=C$$

$$x_1=\frac{r^2}{a}, y_1=\frac{r^2}{b}, z_1=\frac{r_2^{\,2}}{c}$$

But the point (x_1, y_1, z_1) is on the sphere (1)

$$\therefore \quad x_1^2+y_1^2+z_1^{\,2}=r^2$$

$$\Rightarrow \quad \frac{r^4}{a^2}+\frac{r^4}{b^2}+\frac{r^4}{c^2}=r^2$$

$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}$ Hence proved

Problem (2):- Find the equation to the sphere through the circle $x^2+y^2+z^2=9$, $2x+3y+4z=5$ and the point (1,2,3)

Solution:- The given eq^n. of circle is

$$x^2+y^2+z^2=9$$

The eq^n. of any sphere through the given circle

$$x^2+y^2+z^2-9+K(2x+3y+4z-5)=0 \qquad \text{—(1)}$$

where K is arbitrary.

It passes through the point (1,2,3).

Then

$$
\begin{aligned}
&1^2+2^2+3^2-9+K(2.1+3.2+4.3-5)=0\\
&1+4+9-9+K(2+6+12-5)=0\\
&1+4+K(8+12-5)=0\\
&5+K15=0\\
&15K=-5\\
&K=-5/15=-1/3
\end{aligned}
$$

Substituting the value of K in (1), we get

$$x^2+y^2+z^2-9+-\frac{1}{3}(2x+3y+4z-5)=0$$

$$\Rightarrow x^2+y^2+z^2-9-\frac{2}{3}x-\frac{3}{3}y-\frac{4}{3}z+\frac{5}{3}=0$$

$$\Rightarrow x^2+y^2+z^2-\frac{2}{3}x-y-\frac{4}{3}z-9+\frac{5}{3}=0$$

$$\Rightarrow 3x^2+3y^2+3z^2-2x-3y-4z-27+5=0$$

$$\Rightarrow 3x^2+3y^2+3z^2-2x-3y-4z-22=0$$

Which is the required eq^n. of the sphere.

Problem (3):- A plane passes through a fixed point (f,g,h) and cuts the axes in A,B,C. Show that the locus of the centre of the sphere OABC is $\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=2.$

Solution:- Let OA = α

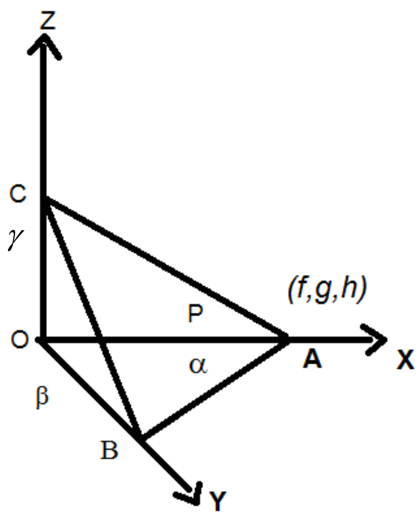
OB = β

OC = γ

Then the eq^n. of plane

ABC is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \quad —(1)$$



Since (1) passes through (f,g,h)

$$\therefore \frac{f}{\alpha}+\frac{g}{\beta}+\frac{h}{\gamma}=1 \qquad —(2)$$

More, the co-ordinate of A,B.C are (α, O, O), (O, β, O), (O, O, γ)

$\therefore$ Required of sphere (O, ABC) is

$$x^2+y^2+z^2-\alpha x-\beta y-\gamma z=O \quad —(3)$$

The centre of (3) is $\left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}\right)$

$$\therefore \quad x=\frac{\alpha}{2} \Rightarrow \alpha=2x$$

$$y={}^{\beta}\!/_{2} \Rightarrow \beta=2y$$

and $z={}^{\gamma}\!/_{2} \Rightarrow \gamma=2z$

Putting the values of α, β, γ in equation (2) we get

$$\frac{f}{2x}+\frac{g}{2y}+\frac{h}{2z}=1$$

$$\Rightarrow \quad \frac{f}{x}+\frac{g}{y}+\frac{h}{z}=2$$

Hence proved

Problem (4):- Find the equation of the spheres which pass through the circle $x^2+y^2+z^2=5,\ x+2y+3z=3$and touch the plane $4x+2y=15$

Solution:- The equation of circle is

$$x^2+y^2+z^2=5$$

The equation of sphere through any circle is

$$(x^2+y^2+2^2-5)+K(x+2y+3z-3)=0 —(1)$$

Where K is constant

Now compare the equation (1) with the general equation of sphere

i.e. $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = O$

we get,

$$2u = K \quad \Rightarrow u = K/2$$
$$2v = 2K \quad \Rightarrow v = K$$

and $2w = 3K \Rightarrow w = 3K/2$

and $d = -3k - 5$

clearly, centre of this sphere (1) is

$$(u, v, -w) = \left(-K/2, -K, -\frac{3K}{2}\right)$$

and radius $= \sqrt{u^2 + v^2 + w^2 - d}$

$$r = \sqrt{\frac{K^2}{4} + K^2 + \frac{9K^2}{4}(-3K - 5)}$$

$$\Rightarrow r = \sqrt{K^2 + \frac{4K^2}{4} + 9K^2 + 12K + 20}$$

$$\Rightarrow r = \sqrt{14K^2 + \frac{12K}{4} + 20}$$

Since, sphere (1) touches the plane $4x + 3y = 15$ then $\perp r$ distance from the centre of sphere is equal to radius of sphere.

$\therefore$ Lenght of perpendicular $=$ Radius of sphere.

$$\Rightarrow \quad \frac{4\left(-K/2\right) + 3\left(-K\right) - 15}{\sqrt{4^2 + 3^2}} = \frac{\sqrt{14K^2 + 12K + 20}}{2}$$

$$\Rightarrow \quad \frac{-2K\left(-3K\right) - 15}{\sqrt{25}} = \frac{\sqrt{14K^2 + 12K + 20}}{2}$$

$$\Rightarrow \quad \frac{-5k-15}{5} = \frac{\sqrt{14k^2+12k+20}}{2}$$

$$\Rightarrow \quad -k-3 = \frac{\sqrt{14k^2+12k+20}}{2}$$

Squaring both sides, we get

$$(-k-3)^2 = \left(\frac{\sqrt{14k^2+12k+20}}{2}\right)^2$$

$$\Rightarrow (-k-3)^2 = \frac{1}{4}\left(14k^2+10k+20\right)$$

$$\Rightarrow k^2+9+6k = \frac{14k^2+12k+20}{4}$$

$$\Rightarrow 4\left(k^2+9+6k\right) = 14k^2+12k+20$$

$$\Rightarrow 4k^2+36+24k = 14k^2+12k+20$$

$$\Rightarrow 10k^2-12k-16=0$$

$$\Rightarrow 5k^2-6k-8=0$$

$$\Rightarrow 5k^2+4k-10k-8=0$$

$$\Rightarrow k\left(5k+4\right)-2\left(5k+4\right)=0$$

$$\Rightarrow \left(5k+4\right)\left(k-2\right)=0$$

$$\Rightarrow 5k+4=0 \quad 0 \quad k-2=0$$

$$k = {}^{-4}\!/_{5} \text{ or } k=2$$

Putting the value of k in equation (1), we get

$$x^2+y^2+z^2-5+\left(\frac{-4}{5}\right)\left(x+2y+3z-3\right)=0$$

$$\Rightarrow \quad x^2+y^2+2^2-5-\frac{4x}{5}-\frac{8y}{5}-\frac{12z}{5}+\frac{12}{5}=0$$

$$\Rightarrow \quad \frac{5x^2+5y^2+-25-4x-8y-12-13z+0=0}{5}$$

$$\Rightarrow \quad 5x^2+5y^2+5z^2-4x-8y+12z-13=0$$

and $x^2+y^2+z^2-5+2(x+2y+3z-3)=0$

$$\Rightarrow x^2+y^2+z^2+2x+4y+6z-6-5=0$$

$$\Rightarrow x^2+y^2+z^2+2x+4y+6z-11=0$$

Problem (5):- Find the equation of the sphere through the circle $x^2+y^2+z^2=9$ and $2x+3y+4z=5$.

Solution:- The given circle is

$$x^2+y^2+z^2=9$$

The eq^n. of sphere through any circle is

$$x^2+y^2+z^2-9+\lambda(2x+3y+4z-5)=0 \quad \text{—(1)}$$

Now compare the eq^n. (1) with the general eq^n. of sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

we get

$$\Rightarrow 2u=2\lambda \Rightarrow u=\lambda$$
$$\Rightarrow 2v=3\lambda \Rightarrow v={}^{3}\!/_{2}\,\lambda$$
$$\Rightarrow 2w=4\lambda \Rightarrow w=2\lambda$$

and $d=-5\lambda-9$

The centre of the given sphere is $\left(-\lambda,-\frac{3}{2}\lambda,-2\lambda\right)$

$\therefore$ the centre should be lie on the given plans.

$$2(-\lambda)+3\left(\frac{-3\lambda}{\alpha}\right)+4(-2\lambda)=5$$

$$\Rightarrow -2\lambda-\frac{9\lambda}{\alpha}-8\lambda=5$$

$$\Rightarrow \frac{-4\lambda-9\lambda-16\lambda=5}{2}$$

$$\Rightarrow -29\lambda=10$$

$$\Rightarrow \lambda=\frac{-10}{29}$$

Now substitute the value of λ in eq^n. (1),

we get

$$x^2+y^2+z^2-9+\frac{-10}{29}(2x+3y+4z-5)=0$$

$$\Rightarrow \quad x^2+y^2+z^2-9-\frac{20x}{29}-\frac{30y}{29}-\frac{40z}{29}+\frac{50}{29}=0$$

$$\Rightarrow \frac{29x^2+29y^2+29x^2-261}{29}-20x-30y-40z+50=0$$

$$\Rightarrow 29x^2+29y^2+29z^2-20x-30y-40z-211=0$$

This is the required eq^n. of the sphere.

Problem (6):- Two spheres of radii r_1, and r_2, cut orthogonally prove that the radius of the common circle is

$$\frac{r_1 r_2}{\sqrt{r_1^2+r_1^2}}$$

Solution:- Let the eq^n. of the common circle be

$$x^2+y^2=\alpha^2, Z=O$$

The equation of any sphere through this common circle is

$$x^2+y^2+z^2+2KZ-\alpha^2=0$$

Let the two given spheres through the common circle and of radii r, and r_2 be

$$x^2+y^2+Z^2+2K_1Z-\alpha^2=0$$

$$\& \qquad x^2+y^2+Z^2+2K_2Z-\alpha^2-0$$

$$\therefore \quad r_1^2=K_1^2+a^2 \quad \text{and} \quad r_2^2=K_2^2+a^2$$

Since the spheres (1) and (2) cut orthogonally, therefore, the condition for orthogonality is

$$2K_1K_2=a^2+a^2$$
$$2K_1K_2=2a^2$$

$$\Rightarrow K_1^2 K_2^2 = a^4$$

$$\Rightarrow \left(r_1^2 - a^2\right)\left(r_2^2 - a^2\right) = a^4$$

$$\Rightarrow r_1^2 r_2^2 - a^2\left(r_1^2 + r_2^2\right) = 0$$

$$\Rightarrow a^2\left(r_1^2 + r_2^2\right) = r_1^2 r_2^2$$

$$\Rightarrow a^2 = \frac{r_1^2 r_2^2}{r_1^2 + r_2^2}$$

$$\Rightarrow a = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

Hence, the required result.