E-content- B.Sc. (Mathematics), Degree-1

Subject- Mathematics, Paper-II

Topic-2 Analytical Geometry (3-D)

Name: Dr. Santosh Kumar (CSIR JRF, SRF, Ph.D.)

Guest Faculty, Department of Mathematics,

Patna Science College, Patna-800005

Mobile No. 8210642534

E-Mail-santoshrathore.kumar20@gmail.com

CYLINDER AND RIGHT CIRCULAR CYLINDER

Problems (1):- Find the equation of the cylinder whose generators are parallel to the line $x=\frac{y}{2}=\frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2=1, z=o.$

Solution:- Let (x', y', z') be any point on the surface of the cylinder.

Then the equation of the line through (x', y', z') paralled to $\frac{x}{1}=\frac{y}{\alpha}=\frac{z}{3}$ is

$$\frac{x-x'}{1}=\frac{y-y'}{2}=\frac{z-z'}{3}$$

This line meets Z = O at the point given by

$$\frac{x-x'}{1}=\frac{y-y'}{2}=\frac{o-z'}{3}$$

$$x-x'=\frac{-z'}{3} \text{ and } \frac{y-y'}{\alpha}=\frac{-z'}{3}$$

$$x=x'\frac{-z'}{3} \quad \text{and } y=y'-\frac{2}{3}z'$$

This point $\left(x'-\frac{z'}{3}, y'-\frac{2z'}{3}, 0\right)$ lies on the guiding curue $z^2+2y^2=1$

$$\left(\frac{3x'-z'}{3}\right)^2+2\left(\frac{3y'-2z'}{3}\right)^2=1$$

$$\frac{9x'^2+z'^2-6x'z'}{9}+\frac{2}{9}\left(9y'^2+4z'^2-12y'z'\right)=1$$

$$\Rightarrow 9x'^2+z'^2-6x'z'+18y'^2+8z'^2-24y'z'=9$$

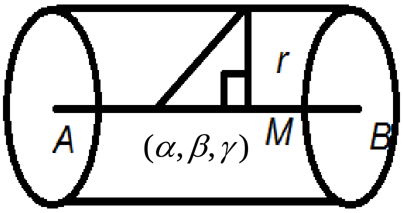
$$\Rightarrow 9x'^2+18y'^2+9z'^2-6x'z'-24y'z'=9$$

tunk the locus of (x', y', z') is

$$-\quad 9x^2+18y^2+9z^2-6xz-24yz=9$$

This is the required eq^n. of the cylinder

Problems (2):- Find the equation of the right circular cylinder, whose radius is ‘r’ and axis is the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$.



Solution:- Let $\rho(x_1,y_1z_1)$ be any point on the surface of the right circular cylinder and if PM be the –r from p to the axis $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ then

. $OP^2-OM^2=PM^2.$ — (1)

PM = the radius of the right circular cylinder = r

The d.r’s of AB are *l,m,n*

Then, the direction casines of AB are

$$\frac{l}{l^2+m^2+n^2},\frac{m}{\sqrt{l^2+m^2+n^2}},\frac{n}{l^2+m^2+n^2}$$

The, axis AB passes through the point (α,β,γ)

Then

$$\text{OP}^2=(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2$$

OM = length of the projection of OP on AB.

OM = $(x_1-\alpha)\frac{\ell}{\ell^2+m^2+n^2}+(y_1-\beta)\frac{m}{\sqrt{\ell^2+m^2+n^2}}+(z_1-\gamma)\frac{n}{\sqrt{\ell^2+m^2+n^2}}$

Then from (1) , me get

$$(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2-\frac{1}{\sqrt{l^2+m^2+n^2}}\begin{bmatrix}(x_1-\alpha)l+(y_1-\beta)m \\ +(z_1-\gamma)n]^2\end{bmatrix}$$

$$=r2$$

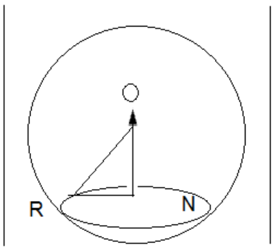
Hence , the locus of $P(x_1,y_1,z_1)$ is

$$(x-\alpha)^2+(y-\beta)^2+(z_1-\gamma)^2\frac{-1}{\sqrt{l^2+m^2+n^2}}\left[(x-\alpha)\ell+(y-\beta)m+(z-\gamma)n\right]^2=r^2$$

This is the required eq^n. of the cylinder.

Problems (3):- Find the equations of the circular cylinder whose guiding circle $x^2+y^2+z^2-9=0,\ x-y+z=3.$

Solution:- The radius of the right circular cylinder is the radius of the given circle and its axis is normal to the plane of the circle through the centre of the circle or sphere.



Since the equation of the sphere is

$$x^2+y^2+z^2=9$$

the coordinator of its centre are (o,o,o) and radius OR = 3

Now, the length of the $\perp r$ ON from the origin on the plane $x-y+z=3$ is

$$\frac{x.o-y.o+z.o-3}{1^2+1^2+1^2}$$

$$=\left|\frac{-3}{\sqrt{3}}\right|$$

$$\Rightarrow\frac{-3}{\sqrt{3}}=\sqrt{3}$$

Thus, OR = 3 and ON = $\sqrt{3}$

$\therefore\quad NR^2=OR^2-ON^2$

= 9 - 3

= 6

$\Rightarrow$ NR = $\sqrt{6}$

Hence, the radius of the right circular cylinder = $\sqrt{6}$

Again, the direction casines of the normal to the circle and thus to the given plans are proportional to 1, -1, 1.

$\therefore$ the eq^n. of the axis which passes through the centre (o,o,o) of the sphere and has direction casines proportional to 1, -1, 1 is

$$\frac{x-o}{1}=\frac{y-o}{-1}=\frac{z-o}{1} \qquad \text{— (1)}$$

$$\Rightarrow \quad \frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$$

Let $P(x,y,z)$ be any point on the cylinder. Then the length of the $\perp$r from P on the line (1) is equal to $\sqrt{6}$.

Also, the square of the $\perp$r distance of P from the line (1)

$$=(x-o)^2+(y-o)^2+(z-o)^2-\left[\frac{1(x-o)-(y-o)+1(z-o)}{1^2+1^2+1^2}\right]$$

$$= x^2+y^2+z^2-\frac{(x-y+z)^2}{3}$$

Thus, $=x^2+y^2+z^2-\dfrac{(x-y+z)^2}{3}=6$

$\Rightarrow 3x^2+3y^2+3z^2-\left(x^2+y^2+z^2-2xy+2xz-2yz\right)$

$= 18$

$\Rightarrow 2x^2+2y^2+2z^2+2xy-2xz+2yz-18=0$

$\Rightarrow x^2+y^2+z^2+xy-xz+yz-9=0$

Which is the required eq^n. of the cylinder.

CONE AND RIGHT CIRCULAR CONE

Problem (1):- Prove that equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=$ represents a cone $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$ it

Solution:- The given equation is

$$ax^2+by^2+cz^2+2ux+2vy+2wz+d=0 \qquad \text{—}(1)$$

Making it homogeneous in $x,y,z,t,$ where t =1,

we get

$$F\left(x,y,z,t\right)\equiv ax^2+by^2+cz^2+\alpha uxt+2vyt+2wzt+d=c$$

The given eq^n. will represent a cone it

$$\frac{\partial F}{\partial x}=0,\quad \frac{\partial \mathrm{F}}{\partial y}=0,\quad \frac{\partial \mathrm{F}}{\partial \mathrm{z}}=0,\quad \frac{\partial \mathrm{F}}{\partial \mathrm{t}}=0, \text{where t=1}$$

These gives

$$2ax+2ut=0 \text{ or } x=-\frac{u}{\alpha}, \text{ as t = 1}$$

$$2by+2vt=0 \text{ or } y=\frac{-v}{a}, \text{ as t =1}$$

$$2cz+2wt=0 \text{ or } z=\frac{-w}{c}, \text{ as t = 1}$$

and

$$2ux+2vy+2wz+2dt=0$$
$$ux+vy+wz+d=0 \text{ as t =1} \qquad \text{—}(2)$$

Substituting the values of x, y, z in (2), we get

$$\frac{u\times -u}{a}+\frac{v\times -v}{a}+\frac{wx-w}{c}+d=0$$

$$\frac{-u^2}{a}+\frac{-v^2}{b}-\frac{w^2}{c}+d=0$$

$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$$

This is the required condition.

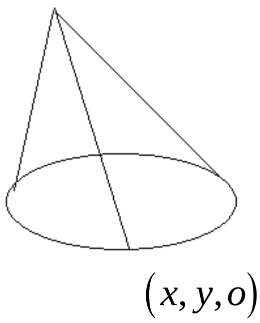
Problem (2):- Find the equation of the cone whose vertex is the point (α, β, v) and whose generators intersect the conic $ax^2 + 2hxy + by^2 + \alpha gx + 2 + y + c = o, z = o$

Solution:- Let the eq^n. of a line through the vertex $v(\alpha, \beta, \gamma)$ of the cone be

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{—(1)}$$

If it meets the plane z =o dt the pt. P then from (1)

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{o-\gamma}{n}.$$



or $x = \frac{\alpha-\ell}{n}\gamma, y = \frac{\beta-m}{n}\gamma$

The point $\left(\frac{\alpha-\ell}{n}, \frac{\beta-m}{n}\gamma, o\right)$

Lie on the guiding curue.

$$ax^2 + 2hxy + by^2 + 2gx + 2ty + c = o, z = o$$

If $a\left(\alpha - \frac{\ell}{n}\gamma\right)^2 + 2h\left(\alpha - \frac{\ell}{n}\gamma\right)\left(\beta - \frac{m}{n}\gamma\right) + b\left(\beta - \frac{m}{n}\gamma\right)^2$

$$+2g\left(\alpha - \frac{\ell}{n}\gamma\right) + 2 + \left(\beta - \frac{m}{n}\gamma\right) + c = o \qquad \text{—(2)}$$

Eliminating l,m,n from (1) & (2), we get the eq^n. of the cone generated by the

line (1) $\frac{\ell}{n} = \frac{x-\alpha}{z-\gamma}$ and $\frac{m}{n} = \frac{y-\beta}{z-\gamma}$

$\therefore \quad \alpha - \frac{\ell}{n}\gamma = \alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma = \frac{z\alpha - x\gamma}{z-\gamma}$

and $\beta - \frac{m}{n}\gamma = \beta - \left(\frac{y-\beta}{z-\gamma}\right)^{\gamma} = \frac{\beta z - \gamma y}{z-\gamma}$

Substituting these values in (2), we get the required eq^n. of the cone is

$$\alpha\left(\frac{\alpha - x\gamma}{z-\gamma}\right)^2 + 2n\left(\frac{z\alpha - x\gamma}{z-\gamma}\right)\left(\frac{\beta z - \gamma y}{z-\gamma}\right) + b\left(\frac{\beta z - \gamma y}{z-\gamma}\right)2 + g\left(\frac{z\alpha - x\gamma}{z-\gamma}\right) + 2 + \left(\frac{\beta z - \gamma y}{z-\gamma}\right) + C = O$$

$$\Rightarrow \alpha\left(\alpha z - x\gamma\right)^2 + 2n\left(z\alpha + x\gamma\right)\left(\beta z - \gamma y\right) + b\left(\beta z - \gamma y\right) + 2g$$
$$\left(z\alpha - x\gamma\right)\left(z-\gamma\right) + 2 + \left(\beta z - \gamma y\right)\left(z - \gamma\right) + C\left(z - v\right)2 = O$$

Problems (3):- Find the condition that the general homogeneous equation of the 2^{nd} degree should represent a right circular cone.

Solution:- The general homogeneous equation of the second degree is

$$\alpha x^2 + by^2 + cz^2 + 2yz + 2g2x + 2hxy = O \qquad \text{— (1)}$$

The equation of the right circular cone whose vertex is the origin is

$$\left(\ell x + my + n_z\right)^2 = \left(\ell^2 + m^2 + n^2\right)\left(x^2 + y^2 + z^2\right)\cos^2\theta.$$

Where θ is the semi-vertical angler of the cone.

Writing $\left(\ell^2 + m^2 + n^2\right)\cos^2\theta = \lambda,$ the equation of the right circular cone becomes

$$\left(\ell^2 + my + nz\right)^2 = \lambda\left(x^2 + y^2 + z^2\right)$$

$$\Rightarrow \ell^2x^2 + m^2y^2 + n^2z^2 + 2\ell mxy + 2mnyz + 2\ell nxz = \lambda x^2 + \lambda y^2 + \lambda z^2$$

$$\Rightarrow \ell^2x^2 - \lambda x^2 + m^2y^2 - \lambda y^2 + n^2z^2 - \lambda z^2 + 2\ell nzx + 2\ell mxy + 2mzyz = O$$

$$\Rightarrow x^2\left(\ell^2 - \lambda\right) + y^2\left(m^2 - \lambda\right) + z^2\left(n^2 - \lambda\right) + 2mnyz + 2n\ell zx + 2\ell xy = O \qquad \text{—(2)}$$

Compeering. (1) & (2) we get

$$\frac{a}{a^2 - \lambda} = \frac{b}{m^2 - \lambda} = \frac{c}{n^2 - \lambda} = \frac{f}{mn} = \frac{g}{n\ell} = \frac{h}{\ell m} = k \text{ (say)}$$

$$\therefore \quad g = nlk, h = lmk \text{ and } f = mnk$$
$$gh = \ell^2 mnk^2$$
$$\frac{gh}{f} = \frac{\ell^2 nk^2}{mnk} = \ell^2 k \qquad \text{— (3)}$$

But from the first equality

$$a = \left(\ell^2 - \lambda\right) K$$

$$i.e.\quad \ell^2 - \lambda = \frac{9}{k} \qquad \text{— (4)}$$

Eliminating ℓ between (3) & (4) we get

$$\frac{gh}{fk} - \lambda = \frac{9}{k}$$

$$\Rightarrow \frac{gh}{fk} - \frac{a}{k} = \lambda$$

$$\Rightarrow \frac{gh - af}{f} = \lambda k$$

Similarly,

$$\frac{hf - bg}{g} = \lambda k \text{ and } \frac{fg - ch}{h} = \lambda k.$$

Thus, the required condition is

$$\frac{gh - af}{f} = \frac{hf - bg}{g} = \frac{fg - ch}{h}$$

Problem (4):- Find the equations to the line in which the plane $2x + y - z = \text{o}$ cuts the cone $4x^2 y^2 + 3z^2 = 0.$

Solution:- Since the vertex of the cone is origin and the plane passes through origin.

Hence, let the eq^n. of the line be

$$\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n} \qquad \text{— (1)}$$

In which the plane $2x + y - z = \text{o}$ wets the cone $4x^2 y^2 + 3z^2 = \text{o}$

$$\therefore\ 4\ell^2 m^2 + 3n^2 = \text{o} \qquad \text{— (2)}$$

and $\quad 2\ell + m - n = \text{o} \qquad \text{— (3)}$

Eliminating 'n' between (2) & (3), we get

$$4\ell^2 - m^2 + 3\left(2\ell + m\right)^2 = o$$

$$\left(2\ell - m\right)\left(2\ell + m\right) + 3\left(2\ell + m\right)^2 = o$$

$$\Rightarrow \ (2\ell+m)\left[2\ell-m+3(2\ell+m)\right]=\text{O}$$
$$\Rightarrow \ (2\ell+m)(4\ell+m)=\text{O}$$
$$\Rightarrow \ 2\ell+m=\text{O}\Rightarrow\frac{\ell}{m}=\frac{-1}{\alpha}$$

and $\quad 4\ell+m=\text{o} \ \Rightarrow \ \frac{\ell}{m}=\frac{1}{4}$

when $\frac{\ell}{m}=\frac{-1}{2}$ then (3) $\Rightarrow n=\text{o}$

$$\therefore \ \frac{\ell}{-1}=\frac{m}{2}=\frac{n}{\text{O}}$$

When $4\ell+m=\text{o}$ then from (3), we have

$$2\ell-n=\text{O}$$

or $\frac{\ell}{-1}=\frac{n}{2}$

$$\therefore \ \frac{\ell}{-1}=\frac{m}{4}=\frac{n}{\alpha}$$

Hence the lines of section are

$$\frac{x}{-1}=\frac{y}{\alpha}=\frac{z}{o} \ ; \ \frac{x}{1}=\frac{y}{4}=\frac{z}{\alpha}$$

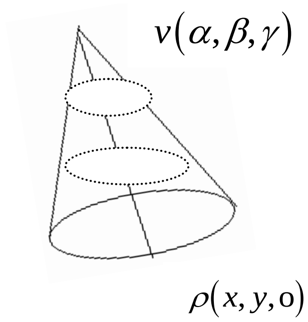
Problem (5):- Find the equation of the cone whose vertex is (α,β,γ) and the guiding curve is the conec $z=o, ax^2+by^2+2hxy+2gx+2fy+czo.$

Solution:- Let the equation of a line through the vertex $V(\alpha,\beta,v)$ of the conc be.

$$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{— (1)}$$

If it meets the plane z = o at the point P then from (1)

$$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{-\gamma}{n}$$



or $\qquad x=\frac{\alpha-\ell\gamma}{n}, y=\beta-\frac{m}{n}\gamma$

The point $\left(\alpha-\frac{\ell}{n}\gamma,\beta\frac{-m\gamma}{n},\text{o}\right)$ will lie on the guiding curve

$$ax+2hxy+by^2+2gx+2fy+c=o,z=o$$

It $a\left(\alpha-\frac{\ell}{n}\gamma\right)^2+2h\left(\alpha-\frac{\ell}{n}\gamma\right)\left(\beta\frac{-m}{n}\gamma\right)+b\left(\beta\frac{-m}{n}\gamma\right)^2$

$$+2g\left(\alpha-\frac{\ell}{n}\gamma\right)+2f\left(\beta-\frac{m}{n}\gamma\right)+\text{c}=\text{o} \qquad \text{— (2)}$$

Elimination, ℓ,m,n from (1) & (2), we get the eq^n. of the cone generated by the line (1)

from (1)

$$\frac{\ell}{n}=\frac{x-\alpha}{z-\gamma} \text{ and } \frac{m}{n}=\frac{y-\beta}{z-\gamma}$$

$$\therefore \quad \alpha\frac{\ell}{n}\gamma=\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma=\frac{\alpha z-x\gamma}{z-\gamma}$$

$$\text{and } \beta=\frac{m\gamma}{n}=\beta-\left(\frac{y-\beta}{z-\gamma}\right)^{\gamma}$$

$$=\frac{\beta z-\beta\gamma-y\gamma+\beta\gamma}{z-r}$$

$$=\frac{\beta z\text{ - }y\gamma}{z-\gamma}$$

Substituting these values in (2), the required eq^n. of the cone is

$$a\left(\frac{az-\gamma x}{z-\gamma}\right)^2+2h\left(\frac{az-\gamma x}{z-\gamma}\right)\left(\frac{\beta z-\gamma y}{z-\gamma}\right)+b\left(\frac{\beta z-\gamma y}{z-\gamma}\right)^2$$

$$+2g\left(\frac{az-\gamma x}{z-\gamma}\right)+2f\left(\frac{\beta z-\gamma y}{z-\gamma}\right)+\ \text{c}=\text{o}$$

$$a\left(az-\gamma x\right)^2+2h\left(az-\gamma x\right)\left(\beta z-\gamma y\right)+b\left(\beta z-\gamma y\right)^2$$

$$+\ 2g\left(az-\gamma x\right)\left(z-\gamma\right)+2f\left(\beta z-\gamma y\right)\left(z-\gamma\right)+C\left(z-\gamma\right)^2=\text{O}$$

Problem (6):- Find the equation of a right circular cone whose vertex is (3,2,1) axis is the line $\frac{x-3}{4}=\frac{y-2}{1}=\frac{z-1}{3}$ and semi-vertical angle is 30:.

Sol^n. Let ℓ, m, n be the direction ratios of any generator of the conc. The its direction cosines are

$$\frac{\ell}{\sqrt{\ell^2+m^2+n^2}},\quad \frac{m}{\ell^2+m^2+n^2},\quad \frac{n}{\sqrt{\ell^2+m^2+n^2}}$$

CONICOID AND PROCEED AS ABOVE PROBLEM

Problem (1):- Find the condition that two diameters of an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ should be conjugate.

Solution:- Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the extremities of two conjugate diameters

$$\frac{x}{\ell_1}=\frac{y}{m_1}=\frac{z}{n_1} \qquad \text{— (1)}$$

and $$\frac{x}{\ell_2}=\frac{y}{m_2}=\frac{z}{n_2} \qquad \text{— (2)}$$

of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

The diametral plane of OP is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=\text{o}$$

Since $Q(x_2, y_2, z_2)$ lies on it, therefore

$$\frac{x_1x_2}{a_2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=o \qquad \text{— (3)}$$

Again since (1) and (2) pass respectively through $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

$$\therefore\ \frac{x_1}{\ell_1}=\frac{y_1}{m_1}=\frac{z_1}{n_1} \text{ and } \frac{x_2}{\ell_2}=\frac{y_2}{m_2}=\frac{z_2}{h_2}$$

Hence from (3), we get the required condition as

$$\frac{\ell_1\ell_2}{a^2}+\frac{m_1m_2}{b^2}+\frac{n_1m_2}{c^2}=\text{o}$$

Problem (2):- For the ellipsoid $2x^2+3y^2+4z^2=1$, find the equation of the tangent plane which is parallel to $x+y+z=3$

Solution:- The equation of any plane parallel to $x+y+z=3$ is $x+y+z=k$ — (1)

Now, we have to find the value of k, so that the plane $x+y+z=k$ becomes tangent plane to the ellipsoid $2x^2+3y^2+4z^2=1$

Let $x+y+z=k$ touch the ellipsoid at the point (x_1, y_1, z_1) so that the equation of the tangent plane at (x_1, y_1, z_1) is $2xx_1+3yy_1+4zz_1=1$ — (2)

Comparing (1) & (2) , we have $\frac{2x_1}{1}=\frac{3y_1}{1}=\frac{4z_1}{1}=k$

$$\Rightarrow x_1=\frac{1}{2k}, y_1=\frac{1}{3k}, \ z_1=\frac{1}{4k} \text{ — (3)}$$

Since, the point (x_1, y_1, z_1) lies on the ellipsoid $2x^2+3y^2+4z^2=1$

$$\therefore \quad 2x_1^2+3y_1^2+4z_1^2=1$$

Hence from (3)

$$2.\frac{1}{4k^2}+3.\frac{1}{9k^2}+4.\frac{1}{16k^2}=1$$

$$\Rightarrow \frac{1}{k^2}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=1$$

$$\Rightarrow k^2=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

$$\Rightarrow k^2=\frac{6+4+3}{12}$$

$$\Rightarrow k^2=\frac{13}{12}$$

$$\Rightarrow k=\pm\sqrt{13/12}$$

Hence, the required equation of the tangent planes are.

$$x+y+z=\pm\sqrt{13/12} \quad \text{Ans}$$

Problem (3):- To find the condition that the plane $\ell x+my+nz=p$ should touch the central concoid $ax^2+by^2+cz^2=1$

Solution:- Let the plane $\ell x+my+nz=p$ —(1)

touch the central conicoid

$$ax^2+by^2+cz^2=1 \text{ at the point } (\alpha,\beta,\gamma)$$

We know that the equation of the tangent plane to the central conicoid at the point (α,β,γ) is

$$a\alpha x+b\beta y+c\gamma z=1 \qquad \text{— (2)}$$

Since the equation (1) & (2) represent the same tangent plane, hence they are identical Therefore. Comparing (1) & (2), we get

$$\frac{a\alpha}{\ell}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{1}{p}$$

$$\therefore\quad \alpha=\frac{\ell}{aP},\quad \beta=\frac{m}{b\beta},\quad \gamma=\frac{n}{cp}$$

Now, since the point (α,β,γ) lies on the conicoid

$$ax^2+by^2+cz^2=1$$

$$\therefore\quad a\alpha^2+b\beta^2+c\gamma^2=1$$

$$\Rightarrow\quad \frac{\ell^2}{ap^2}+\frac{m^2}{bp^2}+\frac{n^2}{cp^2}=1 \qquad \text{[from (3)]}$$

$$\Rightarrow\quad \frac{\ell^2}{ap^2}+\frac{m^2}{bp^2}+\frac{n^2}{cp^2}=1$$

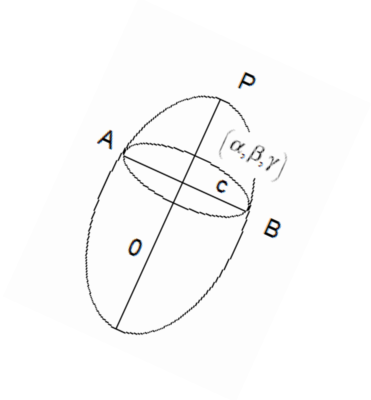
$$\therefore\quad \frac{\ell^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

Which is the required condition

Problem (4):- Find the locus of chords of the conicoid $ax^2+by^2+cz^2=1$ which are bisected at (α,β,γ) (2014) [12.(a)]

Solution:- Let $\dfrac{x-\alpha}{\ell}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}=r$ be the equation of any chord AB of the conicoid $ax^2+by^2+cz^2=1$ — (2)

passing through (α, β, γ).



The co-ordinate of any point on (1) are

$(\alpha + \ell r, \beta + mr, \gamma + nr)$

If it lies on the conicoid, then its co-ordinate must be satisfied the eq^n. (2)

Hence $a(\alpha + \ell r)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2 = 1$

$\Rightarrow a(\alpha^2 + \ell^2 r^2 + 2\alpha\ell r) + b(\beta^2 + m^2 r^2 + 2\beta mr) + c(r^2 + n^2 r^2 + 2\gamma nr) = 1$

$\Rightarrow a\alpha^2 + b\beta^2 + c\gamma^2 + a\ell^2 r^2 + bm^2 r^2 + cn^2 r^2 + 2a\alpha\ell r + 2b\beta mr + 2c\gamma nr = 1$

$\Rightarrow r^2(a\ell^2 + bm^2 + cn^2) + 2r(a\alpha\ell + b\beta m + c\gamma n) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0$

This is a quadratic equation in r. Hence it will have two roots. Let the two roots be r_1 and r_2. Geometrically, it means that any line passing through (α, β, γ) will out the conicoid in two points.

If C is the middle point of the chord AB then $r_1 + r_2 = O$

$\therefore$ we have

$a\alpha\ell + b\beta m + c\gamma n = O$ —— (3)

Hence ℓ, m, n are variables.

Hence, the required locus of chords besecled at (α, β, γ) is obtained by eliminating ℓ, m, n from (1) and (3)

Thus, we have

$a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma) = O$

$\Rightarrow a\alpha x - a\alpha^2 + b\beta y - b\beta^2 + c\gamma z - c\gamma^2 = O$

$\Rightarrow a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2$ —— (4)

This is the eq^n. of the plane which can be written as $S_1 = T$

Where S_1 and T have usual meanings since all chords of the conicoid passing there (α, β, γ) are biscted there, C is the centre of this plane.

Problem (5):- Find the locus of the equal conjugate diameter of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution:- Let $P(x_1, y_1, z_1)$, $Q\ (x_2 y_2, z_2)$ and R (x_3, y_3, z_3) be the extremities of any set of three equal conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{— (1)}$$

Let ℓ, m, n be the direction cosines and r be the length of any one of the equal conjugate diameters.

Then its equation is $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}=r$ — (2)

Where $r^2 = x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2 = x_3^2 + y_3^2 + z_3^2$

$\therefore$ Adding them

$$3r^2 = \left(x_1^2 + x_2^2 + x_3^2\right) + \sum y_1^2 + \sum z_1^2$$

$$3r^2 = a^2 + b^2 + c^2 \qquad \text{— (3)}$$

Now, from (2), the point $(\ell r, mr, nr)$ lies on the ellipsoid (1)

$$\therefore \frac{(\ell r)^2}{a^2}+\frac{(mr)^2}{b^2}+\frac{(nr)^2}{c^2}=1=\ell^2+m^2+n^2$$

$$\Rightarrow \frac{\ell^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}=\frac{\ell^2+m^2+n^2}{r^2}$$

$$\Rightarrow \frac{\ell^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}=\frac{\left(\ell^2+m^2+n^2\right)}{\left(a^2+b^2+c^2\right)}\text{[from (3)]} \qquad \text{— (4)}$$

This relation is true for any set of equal conjugate diameters of the ellipsoid. Hence eliminating ℓ, m, n between (2) and (4), the required locus is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) = \frac{3\left(x^2+y^2+z^2\right)}{\left(a^2+b^2+c^2\right)}$$

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\left(a^2+b^2+c^2\right)=3\left(x^2+y^2+z^2\right)$$

Which after simplification becomes

$$x^2+x^2\frac{b^2}{a^2}+x^2\frac{c^2}{a^2}+y^2\frac{a^2}{b^2}+y^2+z^2\frac{c^2}{b^2}+x^2\frac{c^2}{a^2}+y\frac{c^2}{b^2}+z^2=3x^2+3y^2+3z^2$$

$$\Rightarrow x^2\left(a+\frac{b^2}{a^2}+\frac{c^2}{a^2}-3\right)+y^2\left(1+\frac{a^2}{b^2}+\frac{c^2}{b^2}-3\right)+z^2\left(1+\frac{c^2}{a^2}+\frac{c^2}{b^2}-3\right)=0$$

$$\Rightarrow \frac{x^2}{a^2}\left(-2a^2+b^2+c^2\right)+\frac{y^2}{b^2}\left(-2b^2+a^2+c^2\right)+\frac{z^2}{c^2}\left(-2c^2+b^2+a^2\right)=0$$

$$\Rightarrow -\left[\left(2a^2-b^2-c^2\right)\frac{x^2}{a^2}+\left(2b^2-a^2-c^2\right)\frac{y^2}{b^2}+\frac{z^2}{c^2}\left(2c^2-b^2-a^2\right)\right]$$

$$\Rightarrow \left(2a^2-b^2-c^2\right)\frac{x^2}{a^2}+\left(2b^2-a^2-c^2\right)\frac{y^2}{b^2}+\left(2c^2-b^2-a^2\right)\frac{z^2}{c^2}=0$$

and this is a cone generated by the equal conjugate diameters.

Problem (6):- If $(x_1, y_1, z_1),(x_2, y_2, z_2)$ and (x_3, y_3, z_3) be the coordinates of the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ then show that

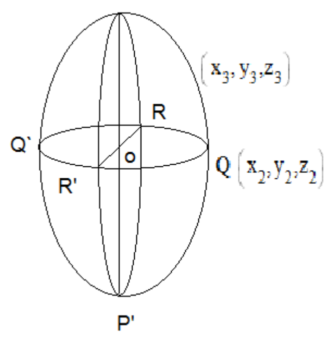
$$\sum x_1^2=a^2;\sum y_1^2=b^2;\sum z_1^2=c^2$$

$$\sum x_1y_1=0\ ;\ \sum y_1z_1 0\ ;\ \sum z_1z_1=0$$

Sol^n. The equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad — (1)$$

$p(x_1, y_1, z_1)$



Its centre is the origin $0(o,o,o)$

Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid.

There he diametral plane of OP, that is, lane bisecting chord pauallel to OP is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz1}{c^2}=0 \quad \text{— (2)}$$

Let $Q(x_2, y_2, z_2)$ be any point on the section of the ellipsoid by the diametral plane of OP.

Then $Q(x_2, y_2, z_2)$ must satisfy (2)

$$\therefore \quad \frac{x_2x_1}{a^2} + \frac{y_2y_1}{b^2} + \frac{z_2z_1}{c^2}=0 \quad \text{— (3)}$$

This shows that the point $P(x_1, y_1, z_1)$ lies on the dimetral plane

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}=0 \text{ of OQ.}$$

This, if the diametral plane OP passes through Q then the diametral plane of OQ also passes through P.

Let OR be the line of intersection of the diametral planes of OP and OQ where the point $R(x_3, y_3, z_3)$ is on the ellipsoid.

Since the diametral plane (2) of OP passes through $R(x_3, y_3, z_3)$ therefore,

$$\frac{x_3x_1}{a^2} + \frac{y_3y_1}{b^2} + \frac{z_3z_1}{c^2}=0 \quad \text{— (4)}$$

This shows that the diametral plane

$$\frac{xx_3}{a^2} + \frac{yy_3}{b^2} + \frac{zz_3}{c^2}=0 \quad \text{— (5)}$$

of OR must pass through $P(x_1, y_1, z_1)$ similarly, the diameteral plane (5) of OR must also pass through Q (x_2, y_2, z_2)

$$\therefore \quad \frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2}=0 \quad \text{— (6)}$$

Hence the diametral plane of OR is the plane POQ.

We find that the three semi-diameters OP, OQ, OR are such that the diamteral plane of any one of them contains the other tuw.

Hence, OP, OQ, OR are called conjugate semi-diameters of the ellipsoid.

Since the points $P(x_1, y_1, z_1)$, $Q\ (x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ are on the ellipsoid (1) therefore

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \qquad \text{— (7)}$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} = 1 \qquad \text{— (8)}$$

$$\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} + \frac{z_3^2}{c^2} = 1 \qquad \text{— (9)}$$

From (7), (8), (9), we infer that

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c} \ ; \ \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c} \ ; \ \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}$$

are direction cosinis of three straight lines, by virtue of (3), (4), (6) are inutually perpendicular.

Hence from the properties of the direction cosines of three mutually perpendicular linis, we have the following six relations:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{a^2} = 1$$

i.e. $x_1^2 + x_2^2 + x_3^2 = a^2$

i.e. $\sum \mathrm{x}_1^2 = a^2$ — (10)

$$\frac{y_1^2}{a^2} + \frac{y_2^2}{a^2} + \frac{y_3^2}{a^2} = 1$$

i.e. $y_1^2 + y_2^2 + y_3^2 = a^2 \Rightarrow \sum y_1^2 = b^2$ — (11)

$$\frac{z_1^2}{c^2} + \frac{z_2^2}{c^2} + \frac{z_3^2}{c^2} = 1$$

$$z_1^2 + z_2^2 + z_3^2 = c^2$$

i.e. $\sum z_1^2 = c^2$ — (12)

and $\frac{x_1}{a}\cdot\frac{y_1}{b} + \frac{x_2}{a}\cdot\frac{y_2}{b} + \frac{x_3}{a}\cdot\frac{y_3}{b} = 0$

$$x_1y_1 \neq x_2y_2 + x_3y_3 = 0$$

i.e. $\sum x_1y_1 = 0$ — (13)

$$\frac{y_1}{b}\cdot\frac{z_1}{c} + \frac{y_2\cdot}{b} + \frac{y_3}{b}\cdot\frac{z_3}{c} = 0$$

$$y_1z_1 + y_2z_2 + y_3z_3 - 0$$

i.e. $\sum y_1z_1 = 0$ — (14)

$$\frac{z_1}{c}\cdot\frac{x_1}{a} + \frac{z_2}{c}\cdot\frac{x_2}{a} + \frac{z_3}{c}\cdot\frac{x_3}{a} = 0$$

$$z_1x_1 + z_2x_2 + z_3x_3 = 0$$

i.e. $\sum z_1x_1 = 0$ — (15)

Problem (7):- Find the condition that the plane $\ell x + my + nz = p$ may touch the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution :- The given eq^n. of the ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — (1)

Let the plane $\ell x + my + nz = p$ — (2)

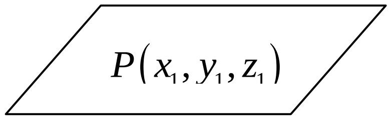
touch the ellipsoid at the point $P(x_1, y_1, z_1)$

then, the plane (2) must be identical with $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1$ — (3)

Hence comparing the coefficients of (2) & (3), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x_1}{\frac{a^2}{\ell}} = \frac{y_1}{\frac{b^2}{m}} = \frac{z_1}{\frac{c^2}{n}} = \frac{1}{p}$$



$\ell x + my + nz = p$

$$x_1 = \frac{a^2\ell}{p}, y_1 = \frac{b^2 m}{p}, z_1 = \frac{c^2 n}{p}$$

Also, since $P(x_1, y_1, z_1)$ is on the ellipsoid (1)

Substituting the values of x_1, y_1 and z_1, we get $\frac{x_1^2}{a^2} + \frac{b^4 m^2}{b^2 p^2} + \frac{c^4 n^2}{c^2 p^2} = 1$

Or $\quad a^2\ell^2 + b^2m^2 + c^2n^2 = p^2$

This is the required condition.

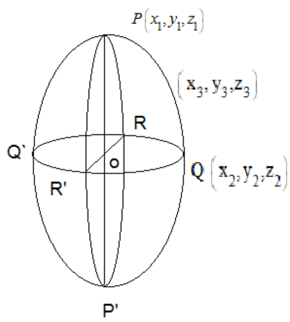
Problem (8):- Define conjugate diameters of an ellipsoid. Prove that the sum of square of any three conjugate semi-diameters of an ellipsoid is a constant.

Definition:- "Conjugate diameters of an ellipsoid:- Any chord that passes through the centre of an ellipse is called its diameter. It follows that the family of parallel chords define two diameters: one in the direction to which they are all parallel and the other locus of their mid points. Such two diameters are called conjugate.

Sol^n.:- The eq^n. of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b} + \frac{z^2}{c^2} = 1 \qquad \text{— (1)}$$

Its centre is the origin O (o,o,o)



Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid. Then the diametral plane of OP, that is the plane bisecting chord parallel to OP is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 0 \qquad \text{—(2)}$$

Let $Q(x_2, y_2, z_2)$ be any point on the section of the ellipsoid by the diametral plane of OP.

Then, $Q(x_2, y_2, z_2)$ must satisfy (2)

$$\frac{x_2 x_1}{a^2}+\frac{y_2 y_1}{b^2}+\frac{z_2 z_1}{c^2}=0 \qquad \text{— (3)}$$

This shows that the point $P(x_1, y_1, z_1)$ lies on the diametral plane

$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}=$ 0 of OQ.

Thus, if the diametral of OP passes through Q than the diametral plane of OQ also passes through P.

Let OR be the line of intersection of the diametral planes of OP and OQ, where the point $R(x_3, y_3, z_3)$ is on the ellipsoid.

Since, the diametral plane (2) of OP passes through $R(x_3, y_3, z_3)$.

$$\therefore \quad \frac{x_3 x_1}{a^2}+\frac{y_3 y_1}{b^2}+\frac{z_3 z_1}{c^2}=0 \qquad \text{— (4)}$$

This shows that the diametral plane $\frac{xx_3}{a^2}+\frac{yy_3+}{b^2}\frac{zz_3}{c^2}=0$ — (5)

of OR must pass through $P(x_1, y_1, z_1)$.

Similarly, the diametral plane (5) of OR must also pass through $Q(x_2, y_2, z_2)$.

$$\therefore \quad \frac{x_2 x_3}{a^2}+\frac{y_2 y_3}{b^2}+\frac{z_2 z_3}{c^2}=0 \qquad \text{— (6)}$$

Hence, the diametral plane of Or is the plane POQ.

We find that the three semi-diameters OP, OQ, OR are such that the diametral plane of any one of them contains the other two.

Hence OP, OQ, OR are called the conjugate semi-diameters of the ellipsoid.

Since, the pts $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ and R(x_3, y_3, z_3) and on the ellipsoid (1)

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1 \qquad \text{— (7)}$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1 \qquad \text{— (8)}$$

$$\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1 \qquad \text{— (9)}$$

From (7), (8), (9), we inter that $\frac{x_1}{a},\frac{y_1}{b},\frac{z_1}{c};\frac{x_2}{a},\frac{y_2}{b},\frac{z_2}{c};\frac{x_3}{a},\frac{y_3}{b},\frac{z_3}{c}$ are the direction ratios of three straight line which are mutually $\perp$r by virtue of (5), (4), (6)

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{x_2^2}{a^2}+\frac{x_3^2}{a^2}=1 \Rightarrow x_1^2+x_2^2+x_3^2=a^2 \Rightarrow \sum x_1^2=a^2 \qquad \text{— (10)}$$

$$\frac{y_1^2}{b^2}+\frac{y_2^2}{b^2}+\frac{y_3^2}{c^2}=1 \Rightarrow y_1^2+y_2^2+y_3^2=a^2 \Rightarrow \sum y_1^2=a^2 \qquad \text{— (11)}$$

$$\frac{z_1^2}{c^2}+\frac{z_2^2}{c^2}+\frac{z_3^2}{c^2}=1 \Rightarrow z_1^2+z_2^2+z_3^2=c^2 \Rightarrow \sum z_1^2=a^2 \qquad \text{— (12)}$$

The sum of the squares of three conjugate semi-diameters $=OP^2+OQ^2+OR^2$

$$=\left(x_1^2+y_1^2+z_1^2\right)+\left(x_2^2+y_2^2+z_2^2\right)+\left(x_3^2+y_3^2+z_3^2\right)$$

$$=\left(x_1^2+x_2^2+x_3^2\right)+\left(y_1^2+y_2^2+y_3^2\right)+\left(z_1^2+z_2^2+z_3^2\right)$$

$$=a^2+b^2+c^2 \qquad \left[\text{from (10, (11) \& (12)}\right]$$

$=$ a constant

Straight lines, Planes, Coplaner, Direction casines, shortest distance

Condition for the homogenous second degree equation $ax^2+by^2+cz^2+2gzx+2hxy+2fyz=0$ to represent a pair of planes. (2015) [10.(a)]

Solution:- The most general homogeneous form of the equation of the second degree in x,y,z is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ — (1)

Since the eq^n. (1) is of 2nd degree homogenous in x,y,z the equations of the plane represented by (1) can be taken as

$$\ell_1 x+m_1 y+n_1 z=0 \qquad \text{— (2)}$$

and $\qquad \ell_2 x + m_2 y + n_2 z = 0$ — (3)

$\therefore$ (1) will represent a pair of planes if it is identically equivalent to

$$\left(\ell_1 x + m_1 y + n_1 z\right)\left(\ell_2 x + m_2 y + n_2 z\right) = 0 \quad \text{— (4)}$$

By equating coefficients in (1) & (4) we get

$$\ell_1 \ell_2 = a, \quad m_1 m_2 = b, \quad n_1 n_2 = C$$

$$m_1 n_2 + m_2 n_1 = 2f, \quad n_1 \ell_2 + n_2 \ell = 2g,$$

$$\ell_1 m_2 + m_1 \ell_2 = 2h.$$

By eliminating $\ell_1, m_1, n_1;\ \ell_2, m_2, n_2$ from the above six relations, we get the required condition.

Now, consider the product of two null determinates.

$$\begin{vmatrix} \ell_1 & \ell_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} \ell_2 & \ell_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0$$

or $$\begin{vmatrix} 2\ell_1\ell_2 & \ell_1 m_2 + \ell_2 m_1 & \ell_1 n_2 + \ell_2 n_1 \\ \ell_2 m_1 + \ell_1 m_2 & 2m_1 m_2 & m_1 n_2 + m_2 n_1 \\ n_1 \ell_2 + n_2 \ell_1 & n_1 m_2 + n_2 m_1 & .\ 2n_1 n_2 \end{vmatrix} = 0$$

or $$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

or $$8\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

This is the required condition in the determinant form.

Expanding this determinant, we get

$$a \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \begin{vmatrix} h & f \\ g & c \end{vmatrix} + g \begin{vmatrix} h & b \\ g & f \end{vmatrix} = 0$$

or $a\left(bc-f^2\right)-h\left(ch-gf\right)+g\left(hf-bg\right)=0$

or $abc-af^2-ch^2+fgh+fgh-bg^2=0$

or $abc+\alpha fgh-af^2-bg^2-ch^2=0$

This is also the required condition.

To find the condition that the two given straight. lines $\frac{x-x_1}{\ell_1}-\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$, $\frac{x_1-x_2}{\ell_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are coplaner.

Solution:- The given eqns. Of straight lines are

$$\frac{x-x_1}{\ell_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \qquad \text{— (1)}$$

and $$\frac{x-x_2}{\ell_3}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \qquad \text{— (2)}$$

The equation of any plane passing through the first line is

$$a\left(x-x_1\right)+b\left(y-y_1\right)+c\left(z-z_1\right)=0 \qquad \text{— (3)}$$

Where $a\ell+bm+cn=0$ — (4)

The plane (3) will contain the line (2) if the point $\left(x_2,y_2,z_2\right)$ lies on it and the line ⊥r to the normal to it.

The condition for this is

$$a\left(x_1-x_2\right)+b\left(y_1-y_2\right)+\left(z_1-z_2\right)=0 \qquad \text{— (5)}$$

and $a\ell_1+bm_1+cn_1=0$ — (6)

Therefore, eliminating a,b,c between (4) (5) & (6) we get.

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ \ell & m & n \\ \ell_1 & m_1 & n_1 \end{vmatrix} = 0 \qquad \text{— (7)}$$

Which is the required condition.

Again eliminating a,b,c between (3), (4) & (5), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \ell & m & n \\ \ell_1 & m_1 & n_1 \end{vmatrix} = 0 \qquad \text{— (8)}$$

Which is the eq^{n}. of the plane containing the two lines.

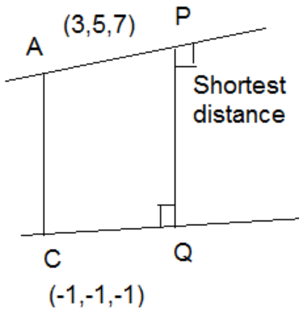
Define skew lines and find the length and equation of line of shortest distance between lines $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}, \frac{x+1}{-7} = \frac{y+1}{-6} = \frac{z+1}{1}$ (2015) [11.(a)]

Let the equation of AB and CD be

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

and

$$\frac{x-1}{-1} = \frac{y-3}{3} = \frac{z-1}{2}$$



The A and C are respectively (3,5,7) and (-1,-1,1). Join A and C. Let (ℓ, m, n) be the direction cosines of the shortest distance PQ between AB and CD.

Then, by the definition of the shortest distance, PQ is ⊥r to both AB and CD.

$$\therefore \quad \ell - 2m + n = 0$$

And $$-7\ell - 6m + n = 0$$

By cross multiplication, we have

$$\frac{\ell}{-2+6}=\frac{M}{-(-1+7)}=\frac{n}{-6-14}$$

$$\frac{\ell}{4}=\frac{M}{-6}=\frac{n}{-20}$$

or $$\frac{\ell}{2}=\frac{M}{-3}=\frac{n}{-10}$$

$$\Rightarrow \frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{4+9+100}}=\frac{1}{\sqrt{113}}$$

$$\therefore \quad \ell=\frac{2}{\sqrt{113}}, M=\frac{-3}{\sqrt{113},}, n=\frac{-10}{\sqrt{113}}$$

(i) The projection of AC on the line PQ is equal to PQ.

$\therefore$ PQ = $(x_2-x_1)\ell+(y_2-y_1)m+(z_2-z_1)n$

$$= (3+1)\frac{.2}{\sqrt{113}}+(5+1)\frac{-3}{\sqrt{113}}+(7+1)\frac{-10}{\sqrt{113}}$$

$$= \frac{8}{\sqrt{113}}-\frac{18}{\sqrt{113}}-\frac{70}{\sqrt{113}}$$

$$=\frac{8-18-70}{\sqrt{113}}$$

$$=\frac{80}{\sqrt{113}}$$

(ii) The eq^n. of the plane containing AB and PQ is

$$\begin{vmatrix} x-3 & y-5 & z-7 \\ 1 & -2 & 1 \\ 2 & -3 & -10 \end{vmatrix}=0$$

or $$(x-3)23-(y-5)-12+(z-7)-1=0$$

$$\Rightarrow 23x-69+12y-60-z+7=0$$

$$\Rightarrow 23x+12y-z-122=0$$

The equation of the plane containing CD and PQ is

$$\begin{vmatrix} x+1 & y+1 & z+1 \\ -7 & -6 & 1 \\ 2 & -3 & -10 \end{vmatrix}=0$$

or $(x+1)63-(y+1)68+(z+1)33=0$

or $63x+63-68y-68+33z+33=0$

or $63x-68y+33z+28=0$

$\therefore$ The equation of the shortest distance are given by

$$23x+12y-z-122=0 \text{ and } 63x-68y+33z+28=0$$

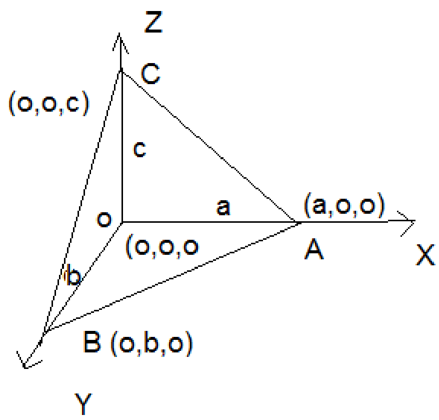
Definition:- Lines which are not parallel and do not interest are called skew liens.

Find the equation of plane which makes intercepts a, b & c on the coordinate axes respectively.

Solution:- Let the eqn. of the plane is $\ell x+my+nz=p$ — (1)

Where ℓ, m, n are the constants.

Let the plane met the co-ordinate axes in the point A,B, and C respect.



Let OA = a

OB = b

OC = C

Then the co-ordinates of the point A are (a,o,o) and those of B and C are (o,b,o) and (o,o,c) respectively.

Since A lies on the plane, therefore on substituting the co-ordinates of A in

(1), we get

$$\ell a = P$$

$$\therefore \ \ell = \frac{P}{a}$$

Similarly, $m=\frac{P}{b}$ and $n=\frac{P}{c}$

Substituting the values of ℓ, m, n in the equations (1), we get

$$\frac{Px}{a}+\frac{P}{b}y+\frac{P}{c}z=P$$

i.e. $$\frac{x}{a}+\frac{y}{b}=\frac{z}{c}=P$$

Which is the required equation of the plane.

A variable plane meets the coordinate axes at A,B, C such that the centroid of the ΔABC is the point (a,b,c). Show that the equations of the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$

Solution:- Let the coordinates of the point A,B,C be respectively $(\alpha,o,o),(o,\beta,o),(o,o,k)$. Therefore, the intercepts of the plane with the co-ordinate axes are (α,β,γ)

Hence, the eq^{n}. of the plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \qquad \text{— (1)}$$

Since (a,b,c) is the centroid of the ΔABC

$$\therefore \quad a=\frac{\alpha}{3} \quad , \quad b=\frac{\beta}{3} \quad , \quad c=\frac{\gamma}{3}$$

i.e. $\alpha=3a \quad , \quad \beta 3b \quad , \quad \gamma 3c$

Substituting the values of in (1), we get

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$$

Which is the required eq^{n}. of the plane.

Show that the lines whose direction cosines are given by $\ell+m+n=0$ and $2mn+3nl-6lm=0$ are $\perp r$ to each other.

Solution:- we have

$$\ell+m+n=0$$

Or $$\ell=-(m+n) \qquad \text{— (1)}$$

Again $$2mn+3n\ell-5\ell m=0$$

or $$2mn-3n(m+n)+5m(m+n)=0 \qquad \text{[from (1)]}$$

or $\quad 2mn+3mn-3n^2+5m^2+5mn=0$

or $\quad 5m^2+4mn-3n^2=0$

$\Rightarrow \quad \frac{5m^2}{n^2}+\frac{4m}{n}-3=0$

This is a quadratic eq^n. in $\frac{m}{n}$. So it must have two values of $\frac{m}{n}$

Let the two values of $\frac{m}{n}$ be $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$

Where $(\ell, m, n,)$ and (ℓ_2, m_2, n_2) are the direction cosines of the given lines.

Then $\quad \frac{m_1}{n_1}.\ \frac{m_2}{n_2} = \frac{-3}{5}$

$$\therefore \quad \frac{m_1 m_2}{3} = \frac{n_1 n_2}{-5} \qquad \text{— (2)}$$

From (1) $\quad m=-(\ell+m)$

Now, $\quad 2mn+3n\ell-5\ell m=0$

or $\quad -\alpha n(\ell+n)+3n\ell+5\ell(\ell+n)=0$

or $\quad 5\ell^2+6\ell n-2n^2=0$

$\Rightarrow \quad 5\frac{\ell^2}{n^2}+6\frac{\ell}{n}-2=0$

The rooks of this eq^n. are $\frac{\ell_1}{n_1}\ \ \frac{\ell_2}{n_2}$.

Then $\frac{\ell_1}{n_1}.\ \frac{\ell_2}{n_2}=-\frac{2}{5}$ or $\ell\ \frac{\ell_2}{2}=\frac{n_1 n_2}{-5}$ — (3)

From (2) & (3), we get

$$\frac{\ell_1\ell_2}{\alpha}=\frac{m_1 m_2}{3}=\frac{n_1 n_2}{-5}$$

$$\Rightarrow \quad \frac{\ell_1\ell_2+m_1 m_2+n_1 n_2}{\alpha+3-5}$$

$$\Rightarrow \quad \ell_1\ell_2+m_1 m_2+n_1 n=0$$

Hence, the given lines we $\perp r$ to each other.

Find the equation of the plane through the point (1,1,1) and $\perp r$ to the planes $x-2y+z=2$ and $4x-3y-z+1=0$.

Solution:- Let the eq^n. of the required plane be $ax+by+cz+d=0$ — (1)

Since it is passing through the point $R(1,1,1)$, we get

$$a.1+b.1+c.1+d=0$$
$$\Rightarrow \quad a+b+c+d=0 \qquad — (2)$$

Again, since the plane (1) is $\perp r$ to the given plane $x-2y+z=2$ and $4x+3y-z+1=0$

$$\therefore \quad aa_1+bb_1+cc_1=0 \qquad \text{(formula)}$$

$$\Rightarrow \quad a.1+b.(-2)+c.1=0$$
$$\Rightarrow \quad a.4+b.3+c.(-1)=0$$

Solving these two eq^n.s by cross multiplication, we get

$$\frac{a}{\alpha-3}=\frac{b}{4+1}=\frac{c}{4+8}$$

$$\frac{a}{1}=\frac{b}{5}=\frac{c}{11}=k \text{ (say) where } k\neq 0$$

$$\therefore \quad a=-k, \quad b=5k, \quad c=11k.$$

Substituting these values in (2), we get

$$-k+5k+11k+d=0$$
or d = - 15 k.

Putting the values of a,b,c & d in (1), we get

$$-kx+5ky+11kz-15k=0$$

$$\Rightarrow x-5y-11z+15=0$$

This is the required eq^n. of the plane.

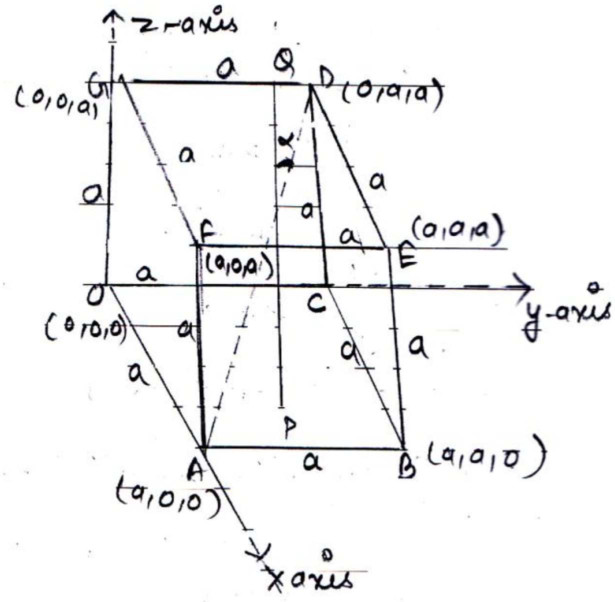
A line makes an angle $\alpha,\beta,\gamma,\delta$ with the diagonal of a cube. Prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=4/3$

Solution:-

Let OABCDEFG be a cube whose each side be a.

Let us take 0 as the origin and OA, OC, OG as x, y, z-axis respectively.

Then, the coordinates of O, A, B, C, D, E, F, G are respectively (o,o,o), (*a*,o,o), (*a*,*a*,o), (o,*a*,o) (o,*a*,*a*), (*a*,*a*,*a*), (*a*,o,*a*), (o,o,*a*).



The diagonal of the cube are AD, BG, CF and OE.

The direction ratios of $AD = x_2 - x_1, y_2 - y_1, z_1 - z_1$

i.e. $o - a,\ a - o,\ a - o$

i.e. $-a,\ \ a,\ \ a$

The direction cosines of AD are

$$\frac{-a}{\sqrt{a^2+a^2+a^2}},\ \frac{a}{\sqrt{a^2+a^2+a^2}},\ \frac{a}{\sqrt{\alpha^2+a^2+a^2}}$$

i.e. $\frac{-\not{a}}{\sqrt{3\not{a}^2}},\ \frac{\not{a}}{\sqrt{3\not{a}^2}},\ \frac{\not{a}}{\sqrt{3\not{a}^2}}$

$= \quad \frac{-1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}$

Let PQ be any line with direction cosines ℓ, m, n which makes angles $\alpha, \beta, \gamma, \delta$ with the diagonal of the cube.

Then $\cos\alpha = \ell_1\ell_2 + m_1m_2 + mn_2$

$$= \frac{1}{\sqrt{3}}(-\ell + m + n)$$

$$\cos\beta = \frac{1}{\sqrt{3}}(\ell - m + n)$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(\ell + m - n)$$

$$\cos\delta = \frac{1}{\sqrt{3}}(\ell + m + n)$$

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\left[(-\ell - m + n)^2 + (\ell - m + n)^2 + (\ell + m - n)^2 + (\ell + m + n)^2\right]$$

$$= \frac{1}{3}\Big[\ell^2 + m^2 + n^2 - 2\cancel{\ell}m + 2\cancel{m}n - 2\cancel{\ell}n + \ell^2 + m^2 + n^2$$
$$2\ell\cancel{m} - 2\cancel{m}n + 2\cancel{\ell}n + \ell^2 + m^2 + n^2 + \cancel{2}\ell m - \cancel{2}mn$$
$$-2\cancel{\ell}n + \ell^2 + m^2 + n^2 + 2\cancel{\ell}m + 2\cancel{m}n + 2\cancel{\ell}n\Big]$$

$$= \frac{1}{3}\left[\left(\ell^2 m^2 + n^2\right) + \left(\ell^2 + m^2 + n^2\right) + \left(\ell^2 + m^2 + n^2\right) + \left(\ell^2 + m^2 + n^2\right)\right]$$

$$= \frac{1}{3}\ (1+1+1+1) \qquad \left[\therefore\ \ \ell^2 + m^2 + n^2 = 1\right]$$

$$= \frac{4}{3}$$

Hence, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

The direction cosines ℓ, m, n of three lines are connected by the relations $\ell + m + n = 0; \alpha\ell n - \alpha mn =$ find them.

Solution:- From the given relations, we have

$$\ell = -(+m + n) \ \ ; \ \ mn - 2\ell(m+n) = 0$$

Eliminating $\ell,$ we get

$$mn + 2(m+n)^2 = 0$$
$$\Rightarrow 2m^2 + 5mn + 2n^2 = 0$$
$$\Rightarrow (m+2n)(2m+n) = 0$$

$$\Rightarrow m + 2n = 0 \quad \text{or} \quad 2m + n = 0 \qquad \text{— (2)} \quad \text{— (1)}$$

Let the two lines whose direction cosines satisfy the given relations, have their direction cosines (ℓ_1, m_1, n_1) and (ℓ_2, m_2, n_2)

Then from (1) $\quad m_1 + 2n_1 = 0$

$$\therefore \quad m_1 = -2n1$$

But $\quad \ell_1 + m_1 + n_1 = 0$

$$\therefore \quad \ell_1 = -(m_1 + n_1)$$
$$= -(-2n_1 + n_1)$$
$$\ell_1 = n_1$$

Also from (2)

$$2m_2 + n_2 = 0$$

$$\therefore n_2 = -2m_2$$

But $\ell_2 + m_2 + n_2 = 0$

$$\ell_2 = -(m_2 + n_2)$$
$$= (m_2 - 2m_2)$$
$$= + m_2$$

But we know that

$$\ell_1^2 + m_1^2 + n_1^2 = 1$$
$$\therefore \quad n_1^2 + 4_1^2 + n_1^2 = 1$$
$$\Rightarrow n_1^2 = \frac{1}{6}$$
$$\Rightarrow n_1 = \frac{1}{\sqrt{6}}$$
$$\text{and} \quad \ell_1 = n_1 = \frac{1}{\sqrt{6}}$$
$$\text{and} \quad m_1 = 2n_1 = \frac{-2}{\sqrt{6}}$$

Hence, $\left(\ell, m_1, n_1\right) = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

Again, $\ell_2^2 + m_2^2 + n_2^2 = 1$

$$m_2^2 + m_2^2 + 4m_2^2 = 1$$

$$\Rightarrow 6m_2^2 = 1$$

$$\Rightarrow m_2 = \frac{1}{\sqrt{6}}$$

$$\therefore \quad \ell_2 = m_2 = \frac{1}{\sqrt{6}}$$

$$n_2 = -2m_2 = \frac{-2}{\sqrt{6}}$$

Hence; $\left(\ell_2, m_2, n_2\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

Now if θ be the angle b/w the two lines, then

$$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$$

$$= \frac{1}{\sqrt{6}}.\frac{1}{\sqrt{6}} + \left(\frac{-2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) + \frac{1}{\sqrt{6}}\left(\frac{-2}{\sqrt{6}}\right)$$

$$= \frac{1}{6} - \frac{2}{6} - \frac{2}{6}$$

$$= \frac{1-4}{6} = \frac{-3}{6}$$

$$\cos\theta = \frac{-1}{2}$$

The acute angle between the two lines is given by

$$\cos\theta = +\frac{1}{2}$$

$$\therefore \quad \theta = \frac{\lambda}{3}$$

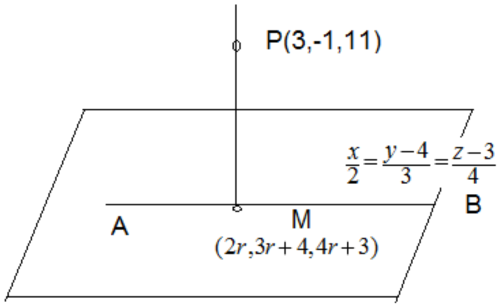
Find the equation of the line which passes through the point $(3,-1,11)$ and is $\perp r$ to the line $\frac{1}{2}x = \frac{1}{3}(x-y) = \quad \neq (z-3)$

Solution:- The given eqn. of lines is

$$\frac{x}{2} = \frac{y-4}{3} = \frac{z-3}{4} = r \text{ (say)}$$

$\therefore \quad x = 2r, y = 3r + 4, z = 4r + 3.$

The coordinates of any point on the line AB are $(2r, 3r+4, 4r+3)$.



Now, the direction cosines of line PM is

$$(2r-3, 3r+5, 4r-8)$$

And it is given that the line through the point (3,-1,11) is $\perp r$ to the given line

$$\therefore \quad 2(2r-3) + 3(3r+5) + 4(4r+3) = 0$$

$$4r - 6 + 9r + 15 + 16r + 12 = 0$$
$$29r + 21 = 0$$
$$r = \frac{-21}{29}$$

Now, the direction cosins of the line PM are

$$\left(2 \times \frac{-21}{29} - 3, 3 \times \frac{-21}{29} + 5, 4 \times \frac{-21}{29} - 8\right)$$
$$= \left(\frac{-129}{29}, \frac{82}{29}, \frac{148}{29}\right)$$

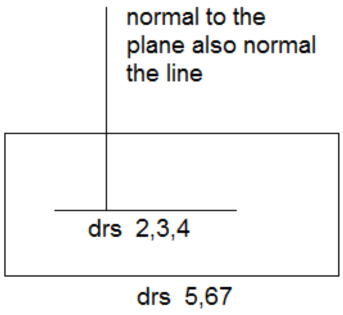
$\therefore$ The required eqn. of the line is

$$\Rightarrow \frac{x-3}{\frac{-129}{219}}, \frac{y+1}{\frac{82}{29}}, \frac{z-11}{\frac{148}{29}}$$

Find the eqn. of the plane through (-1,o,1) containing the line whose d.c's are proportional to (2,3,4) and (5,6,7).

Soln. The eqn. of any plane through the point (-1,o,1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \qquad \text{— (1)}$$
$$\Rightarrow a(x+1) + by + c(z-1) = 0$$



Where a,b,c are dr's of the plane.

Whose value is to be found.

If the plane containing the line where d.c's are proportional to the (2,3,4) and (5,6,7)

$\therefore$ Normal to the plane is also normal to both the line

$\therefore$ By formula

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$
$$2a + 3b + 4c = 0$$
$$5a + 6b + 7c = 0$$

By cross multiplication

$$\frac{a}{21-24} = \frac{b}{20-14} = \frac{c}{12-15}$$
$$\Rightarrow \frac{a}{-3} = \frac{b}{6} = \frac{c}{-3}$$
$$\Rightarrow \frac{a}{-1} = \frac{b}{2} = \frac{c}{-1}$$

Multiplying by -1.

We put the value of a,b,c in equation (1) we get the required eqn. of the plane.

$$-1(x+1) + 2y - 1(z-1) = 0$$
$$\Rightarrow -x - \cancel{1} + 2y + z + \cancel{1} = 0$$
$$\Rightarrow -x + 2y + z = 0$$
$$\Rightarrow \quad x - 2y - z = 0$$

Obtain the condition for given two lines

$$\frac{x-\alpha_1}{\ell_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \text{ and } \frac{x-\alpha_2}{\ell_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$$

to be coplanar. Also find the eqn. of the plane containing then in case they are coplanar.

Solution:- Let the two lines are

$$\frac{x-\alpha_1}{\ell_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \qquad \text{— (1)}$$

and $$\frac{x-\alpha_2}{\ell_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} \quad \text{— (2)}$$

Let the equation of the plane containing the given lines be

$$ax+by+cz+d=0 \quad \text{— (3)}$$

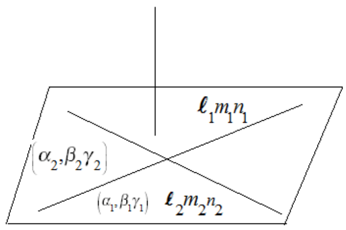
The condition are

$$a\ell_1+bm_1+cn_1=0 \quad \text{— (4)}$$

$$a\alpha_1+b\beta_1+c\gamma_1+d=0 \quad \text{— (5)}$$

and $$a\ell_2+bm_2+cn_2=0 \quad \text{— (6)}$$

$$a\alpha_2+b\beta_2+c\gamma_2=0 \quad \text{— (7)}$$



Subtracting (7) from (8), we get

$$a(\alpha_1-\alpha_2)+b(\beta_1-\beta_2)+c(\gamma_1-\gamma_2)=0 \quad \text{— (8)}$$

Eliminating a,b,c from (8), (4) & (6), we get,

$$\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

This is the required condition

Subtracting (5) from (3), we get

$$a(x-\alpha_1)+b(y-\beta_1)+c(z-\gamma_1)=0 \quad \text{— (9)}$$

Eliminating a,b,c from (9), (4) & (6), we get

$$\begin{vmatrix} x-\alpha_1 & x-\beta_1 & x-\gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

This is the required eqn. of the plane containing the given lines.

Show that the eqn. to the plane containing the line $\frac{y}{b}+\frac{z}{c}=1, x=0$ and parallel to $\frac{x}{a}+\frac{-z}{c}=0, y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0.$ Also prove that if 2d be the shortest distance between three straight line, then $\frac{1}{a^2}-\frac{b}{b^2}-\frac{1}{c^2}=\frac{1}{d^2}$

Solution:- The eqn. of any plane through the first line is

$$\frac{y}{b}+\frac{z}{c}-7+\lambda x=0 \qquad \text{— (1)}$$

The plane (1) will be parallel to the second line if it will be parallel to the plane through the 2nd line.

The equation of any plane through the second line is

$$\frac{x}{a}-\frac{z}{c}-1+\mu y=0 \qquad \text{— (2)}$$

The plane (1) & (2) are parallel if the normals to therm are parallel which is possible only when

$$\frac{\lambda}{1/a} = \frac{1/b}{\mu} = \frac{1/c}{-1/c}$$

$$\therefore \quad \lambda = \frac{-1}{a}, \quad \mu=\frac{-1}{b}$$

Substituting $\lambda=-1/a$ in (1), we get

$$\frac{-x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$$

$$\Rightarrow \quad \frac{x}{a}+\frac{y}{b}-\frac{z}{c}+1=0 \qquad \text{— (3)}$$

Which is the eqn. of the required plane.

The shortest distance between them is the distance of any point on (2) from the plane (3)

$\therefore 2d = \perp r$ from (a,o,o) i.e any pt. on (2) to the plane (3)

$$\not{2}d=\frac{\not{2}}{\sqrt{1/\alpha^2+1/b^2+1/c^2}}$$

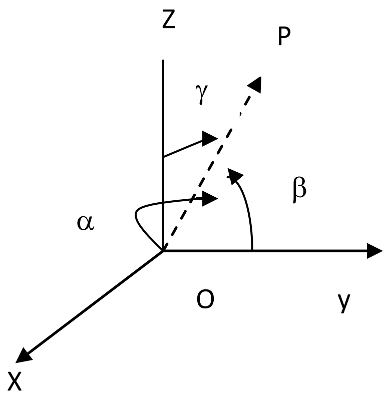
$$d^2 = \frac{1}{1/a^2 + 1/b^2 + 1/c^2}$$

$$\therefore \ \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{d^2}$$

Define direction cosines of a straight line. If ℓ, m, n be direction cosines of a straight line, prove that $\ell^2 + m^2 + n^2 = 1$.

Definition:-

When a line in space is taken in a definite sense from one extreme to the other, the line is said to be directed.

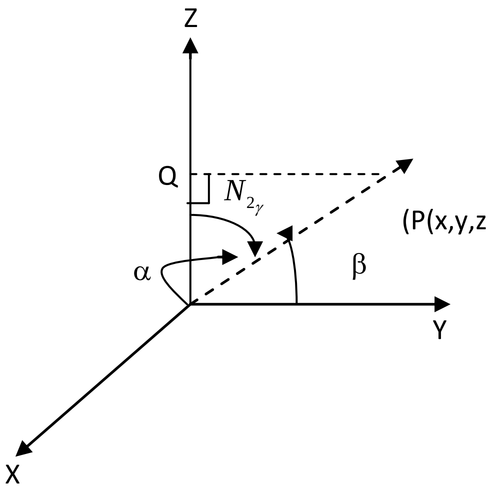


For a directed line OP passing through the origin the angles α, β, γ formed by OP with the $x_1 y_1 z-$ axis respectively are called the direction angles of OP and the cosines of these angles, that is, $\cos\alpha, \cos\beta, \cos\gamma$ are called direction cosines of OP. The direction cosines of a line are generally denoted of a line

by ℓ, m, n.

Then, $\ell = \cos\alpha, m = \cos\beta, n = \cos\gamma.$

Solution:-



Let OP be the directed line through the origin making angles with angles α, β, γ with the axis $x-, y-, z-$ axis respectively.

Consider now a point $P(x, y, z)$ on the directed line.

Let $OP = r$

From P draw $\perp r$ to the Z-axis

Then OQ = Z and $\angle QOP = r$

From right angled ΔQOP

$$\cos\gamma = \frac{OQ}{OP} = \frac{z}{r}$$

$$z = r\cos\gamma.$$

Similarly, drawing $\perp rs$ from P on x- and y-axis we get

$$x = \cos\alpha \text{ and } y = r\cos\beta$$

Now,

$$\frac{OP = r}{\sqrt{x^2 + y^2 + z^2}} = r$$

$$x^2 + y^2 + z^2 = r^2$$

$$r^2\cos^2\alpha + r^2\cos^2\beta + r^2\cos^2\gamma = r^2$$

$$r^2\left(\cos^2 + \alpha\cos^2\beta + \cos^2\gamma\right) = r^2$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

If $\ell = \cos\alpha, m = \cos\beta, n = \cos\gamma$

Then $\ell^2 + m^2 + n^2 = 1$

Transform the Cartesian co-ordinates (3,4,4) of a point to

(i) Spherical polar co-ordinates

(ii) Cylindrical polar co-ordinates

Solution:- (i) Let the cautesian co-ordinates of the point be (x, y, z) and the polar coordinates of the point be (r, θ, ϕ)

Then, $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$

and $z = r\cos\theta$

$\therefore \quad r^2 = x^2 + y^2 + z^2$

$\tan\theta = \dfrac{\sqrt{x^2 + y^2}}{z}$

$\tan\phi = \dfrac{y}{x}$

Here, x = 3, y = 4, z = 5

$$\therefore \quad r^2 = 3^2 + 4^2 + 5^2 = 9 + 16 + 25 = 49$$
$$\therefore r = 7$$

Also, $\tan\theta = \dfrac{\sqrt{x^2 y^2}}{z} = \dfrac{\sqrt{3^2 + 4^2}}{}$

$$= \frac{\sqrt{9+16}}{5}$$
$$= \frac{\sqrt{25}}{5}$$
$$= 5/5$$

$$\tan\theta: \qquad = 1$$

$$\therefore \quad \theta = \bar{\wedge}/4$$

Again, $\tan\phi = y/x = \dfrac{4}{3}$

$$\therefore \quad \phi = \tan^{-1}\left(4/3\right)$$

Hence, the required spherical polar co-ordinates of the given point are $\left(7, \dfrac{\lambda}{4}, \tan^{-1}\left(4/3\right)\right)$.

(ii) Let the cartesian co-ordinates of the point be (x, y, z) and the cylindrical polar coordinates of the point (ζ, ϕ, z)

Then $\alpha = \varsigma\cos\phi, y = \varsigma\sin\phi, z = z.$

$$\therefore \quad \varsigma^2 = x^2 + y^2$$
$$\tan\phi = \frac{y}{x}.$$
$$z = z$$

Here $x = 3, y = 4, z = 5$

$$\therefore \quad \varsigma^2 = 3^2 + 4^2 = 9 + 16 = 25$$
$$\varsigma = 5.$$

Also,

$$\tan\phi = y/x = 4/3$$
$$\phi = \tan^{-1} 4/3$$

and $\quad z = 5.$

Hence, the required cylindrical polar coordinates of the given point are

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