E-content- B.Sc. (Mathematics), Degree-1

Subject- Mathematics, Paper-II

Topic-4 Analytical Geometry (3-D)

Name: Dr. Santosh Kumar (CSIR JRF, SRF, Ph.D.)

Guest Faculty, Department of Mathematics,

Patna Science College, Patna-800005

Mobile No. 8210642534

E-Mail-santoshrathore.kumar20@gmail.com

CONICOID

INTRODUCTION

The general equation of second degree in x, y and z can be put in the form $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$

Just as the general equation of first degree in x, y, z represents a plane surface, similarly the general equation of second degree in x, y and z represents a **quadratic surface** or a **conicoid**. The above equation contains ten constants which can be reduced to nine showing thereby that a quadratic surface or a conicoid can be made to satisfy nine conditions. For example a conicoid can be made to pass through nine given points.

The above equation by transformation of axes can be reduced to certain standard forms each of which represents a surface with a definite name assigned to it. Hence in this chapter we shall discuss some of the central surfaces given below:

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (Ellipsoid)

(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (Hyperboloid of one sheet).

(3) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (Hyperboloid of two sheets).

Problem (1):- Find the condition that two diameters of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ should be conjugate.

Solution:- Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the extremities of two conjugate diameters

$$\frac{x}{\ell_1} = \frac{y}{m_1} = \frac{z}{n_1} \quad \text{— (1)}$$

and

$$\frac{x}{\ell_2} = \frac{y}{m_2} = \frac{z}{n_2} \quad \text{— (2)}$$

of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

The diametral plane of OP is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 0$$

Since $Q(x_2, y_2, z_2)$ lies on it, therefore

$$\frac{x_1x_2}{a_2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2} = o \qquad \text{—(3)}$$

Again since (1) and (2) pass respectively through $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

$$\therefore \quad \frac{x_1}{\ell_1} = \frac{y_1}{m_1} = \frac{z_1}{n_1} \text{ and } \frac{x_2}{\ell_2} = \frac{y_2}{m_2} = \frac{z_2}{h_2}$$

Hence from (3), we get the required condition as

$$\frac{\ell_1\ell_2}{a^2} + \frac{m_1m_2}{b^2} + \frac{n_1n_2}{c^2} = 0$$

Problem (2):- For the ellipsoid $2x^2 + 3y^2 + 4z^2 = 1$, find the equation of the tangent plane which is parallel to $x + y + z = 3$

Solution:- The equation of any plane parallel to $x + y + z = 3$ is $x + y + z = k$ — (1)

Now, we have to find the value of k, so that the plane $x + y + z = k$ becomes tangent plane to the ellipsoid $2x^2 + 3y^2 + 4z^2 = 1$

Let $x + y + z = k$ touch the ellipsoid at the point (x_1, y_1, z_1) so that the equation of the tangent plane at (x_1, y_1, z_1) is $2xx_1 + 3yy_1 + 4zz_1 = 1$ — (2)

Comparing (1) & (2) , we have $\dfrac{2x_1}{1} = \dfrac{3y_1}{1} = \dfrac{4z_1}{1} = k$

$$\Rightarrow x_1 = \frac{1}{2k}, y_1 = \frac{1}{3k}, \quad z_1 = \frac{1}{4k} \text{—(3)}$$

Since, the point (x_1, y_1, z_1) lies on the ellipsoid $2x^2 + 3y^2 + 4z^2 = 1$

$$\therefore \quad 2x_1^2 + 3y_1^2 + 4z_1^2 = 1$$

Hence from (3)

$$2.\frac{1}{4k^2}+3.\frac{1}{9k^2}+4.\frac{1}{16k^2}=1$$

$$\Rightarrow \frac{1}{k^2}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=1$$

$$\Rightarrow k^2=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

$$\Rightarrow k^2=\frac{6+4+3}{12}$$

$$\Rightarrow k^2=\frac{13}{12}$$

$$\Rightarrow k=\pm\sqrt{13/12}$$

Hence, the required equation of the tangent planes are.

$$x+y+z=\pm\sqrt{13/12} \quad \text{Ans.}$$

Problem (3):- To find the condition that the plane $\ell x+my+nz=p$ should touch the central concoid $ax^2+by^2+cz^2=1$

Solution:- Let the plane $\ell x+my+nz=p$ —(1)

touch the central conicoid

$ax^2+by^2+cz^2=1$ at the point (α,β,γ)

We know that the equation of the tangent plane to the central conicoid at the point (α,β,γ) is

$$a\alpha x+b\beta y+c\gamma z=1 \qquad \text{—(2)}$$

Since the equation (1) & (2) represent the same tangent plane, hence they are identical Therefore. Comparing (1) & (2), we get

$$\frac{a\alpha}{\ell}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{1}{p}$$

$$\therefore \quad \alpha=\frac{\ell}{aP}, \quad \beta=\frac{m}{b\beta}, \quad \gamma=\frac{n}{cp}$$

Now, since the point (α,β,γ) lies on the conicoid

$$ax^2+by^2+cz^2=1$$

$$\therefore \quad a\alpha^2+b\beta^2+c\gamma^2=1$$

$$\Rightarrow \frac{\ell^2}{ap^2}+\frac{m^2}{bp^2}+\frac{n^2}{cp^2}=1 \qquad \text{[from (3)]}$$

$$\Rightarrow \frac{\ell^2}{ap^2}+\frac{m^2}{bp^2}+\frac{n^2}{cp^2}=1$$

$$\therefore \quad \frac{\ell^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

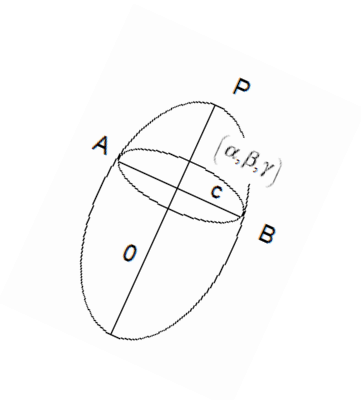
Which is the required condition

Problem (4):- Find the locus of chords of the conicoid $ax^2+by^2+cz^2=1$ which are bisected at (α,β,γ) (2014) [12.(a)]

Solution:- Let $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ be the equation of any chord AB of the conicoid $ax^2+by^2+cz^2=1$

— (2)

passing through (α,β,γ).



The co-ordinate of any point on (1) are

$(\alpha+\ell r,\beta+mr,\gamma+nr)$

If it lies on the conicoid, then its co-ordinate must be satisfied the eq^n. (2)

Hence $a(\alpha+\ell r)^2+b(\beta+mr)^2+c(\gamma+nr)^2=1$

$$\Rightarrow a(\alpha^2+\ell^2r^2+2\alpha\ell r)+b(\beta^2+m^2r^2+2\beta mr)+c(r^2+n^2r^2+2\gamma nr)=1$$

$$\Rightarrow a\alpha^2+b\beta^2+c\gamma^2+a\ell^2r^2+bm^2r^2+cn^2r^2+2a\alpha\ell r+2b\beta mr+2c\gamma nr=1$$

$$\Rightarrow r^2(a\ell^2+bm^2+cn^2)+2r(a\alpha\ell+b\beta m+c\gamma n)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0$$

This is a quadratic equation in r. Hence it will have two roots. Let the two roots be r_1 and r_2. Geometrically, it means that any line passing through (α, β, γ) will out the conicoid in two points.

If C is the middle point of the chord AB then $r_1 + r_2 = O$

$\therefore$ we have

$$a\alpha\ell + b\beta m + c\gamma n = O \qquad —(3)$$

Hence ℓ, m, n are variables.

Hence, the required locus of chords besecled at (α, β, γ) is obtained by eliminating ℓ, m, n from (1) and (3)

Thus, we have

$$a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma) = O$$

$$\Rightarrow a\alpha x - a\alpha^2 + b\beta y - b\beta^2 + c\gamma z - c\gamma^2 = O$$

$$\Rightarrow a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2 \qquad —(4)$$

This is the eq^n. of the plane which can be written as $S_1 = T$

Where S_1 and T have usual meanings since all chords of the conicoid passing there (α, β, γ) are biscted there, C is the centre of this plane.

Problem (5):- Find the locus of the equal conjugate diameter of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution:- Let $P(x_1, y_1, z_1)$, $Q\ (x_2 y_2, z_2)$ and R (x_3, y_3, z_3) be the extremities of any set of three equal conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad —(1)$$

Let ℓ, m, n be the direction cosines and r be the length of any one of the equal conjugate diameters.

Then its equation is $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n} = r$ — (2)

Where $r^2 = x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2 = x_3^2 + y_3^2 + z_3^2$

$\therefore$ Adding them

$$3r^2 = \left(x_1^2 + x_2^2 + x_3^2\right) + \sum y_1^2 + \sum z_1^2$$

$$3r^2 = a^2 + b^2 + c^2 \qquad \text{— (3)}$$

Now, from (2), the point $(\ell r, mr, nr)$ lies on the ellipsoid (1)

$$\therefore \frac{(\ell r)^2}{a^2} + \frac{(mr)^2}{b^2} + \frac{(nr)^2}{c^2} = 1 = \ell^2 + m^2 + n^2$$

$$\Rightarrow \frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = \frac{\ell^2 + m^2 + n^2}{r^2}$$

$$\Rightarrow \frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = \frac{\left(\ell^2 + m^2 + n^2\right)}{\left(a^2 + b^2 + c^2\right)} \text{[from (3)]} \qquad \text{— (4)}$$

This relation is true for any set of equal conjugate diameters of the ellipsoid. Hence eliminating ℓ, m, n between (2) and (4), the required locus is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = \frac{3\left(x^2 + y^2 + z^2\right)}{\left(a^2 + b^2 + c^2\right)}$$

$$\Rightarrow \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(a^2 + b^2 + c^2\right) = 3\left(x^2 + y^2 + z^2\right)$$

Which after simplification becomes

$$x^2 + x^2\frac{b^2}{a^2} + x^2\frac{c^2}{a^2} + y^2\frac{a^2}{b^2} + y^2 + z^2\frac{c^2}{b^2} + x^2\frac{c^2}{a^2} + y\frac{c^2}{b^2} + z^2 = 3x^2 + 3y^2 + 3z^2$$

$$\Rightarrow x^2\left(a + \frac{b^2}{a^2} + \frac{c^2}{a^2} - 3\right) + y^2\left(1 + \frac{a^2}{b^2} + \frac{c^2}{b^2} - 3\right) + z^2\left(1 + \frac{c^2}{a^2} + \frac{c^2}{b^2} - 3\right) = 0$$

$$\Rightarrow \frac{x^2}{a^2}\left(-2a^2 + b^2 + c^2\right) + \frac{y^2}{b^2}\left(-2b^2 + a^2 + c^2\right) + \frac{z^2}{c^2}\left(-2c^2 + b^2 + a^2\right) = 0$$

$$\Rightarrow -\left[\left(2a^2-b^2-c^2\right)\frac{x^2}{a^2}+\left(2b^2-a^2-c^2\right)\frac{y^2}{b^2}+\frac{z^2}{c^2}\left(2c^2-b^2-a^2\right)\right]$$

$$\Rightarrow \left(2a^2-b^2-c^2\right)\frac{x^2}{a^2}+\left(2b^2-a^2-c^2\right)\frac{y^2}{b^2}+\left(2c^2-b^2-a^2\right)\frac{z^2}{c^2}=0$$

and this is a cone generated by the equal conjugate diameters.

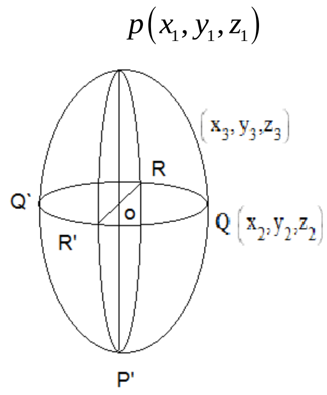
Problem (6):- If $(x_1,y_1,z_1),(x_2,y_2,z_2)$ and (x_3,y_3,z_3) be the coordinates of the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ then show that

$$\sum x_1^2=a^2;\sum y_1^2=b^2;\sum z_1^2=c^2$$

$$\sum x_1y_1=0\ ;\ \sum y_1z_10\ ;\ \sum z_1z_1=0$$

Sol^n. The equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{— (1)}$$



Its centre is the origin $0(o,o,o)$

Let $P(x_1,y_1,z_1)$ be any point on the ellipsoid.

There he diametral plane of OP, that is, lane bisecting chord pauallel to OP is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz1}{c^2}=0 \qquad \text{— (2)}$$

Let $Q(x_2,y_2,z_2)$ be any point on the section of the ellipsoid by the diametral plane of OP.

Then $Q(x_2,y_2,z_2)$ must satisfy (2)

$$\therefore\ \frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2}=0 \qquad \text{— (3)}$$

This shows that the point $P(x_1,y_1,z_1)$ lies on the dimetral plane

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}=0 \text{ of OQ.}$$

This, if the diametral plane OP passes through Q then the diametral plane of OQ also passes through P.

Let OR be the line of intersection of the diametral planes of OP and OQ where the point $R(x_3, y_3, z_3)$ is on the ellipsoid.

Since the diametral plane (2) of OP passes through $R(x_3, y_3, z_3)$ therefore,

$$\frac{x_3x_1}{a^2} + \frac{y_3y_1}{b^2} + \frac{z_3z_1}{c^2} = 0 \qquad \text{— (4)}$$

This shows that the diametral plane

$$\frac{xx_3}{a^2} + \frac{yy_3}{b^2} + \frac{zz_3}{c^2} = 0 \qquad \text{— (5)}$$

of OR must pass through $P(x_1, y_1, z_1)$ similarly, the diameteral plane (5) of OR must also pass through Q (x_2, y_2, z_2)

$$\therefore \quad \frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2} = 0 \qquad \text{— (6)}$$

Hence the diametral plane of OR is the plane POQ.

We find that the three semi-diameters OP, OQ, OR are such that the diamteral plane of any one of them contains the other tuw.

Hence, OP, OQ, OR are called conjugate semi-diameters of the ellipsoid.

Since the points $P(x_1, y_1, z_1)$, $Q\ (x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ are on the ellipsoid (1) therefore

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \qquad \text{— (7)}$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} = 1 \qquad \text{— (8)}$$

$$\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} + \frac{z_3^2}{c^2} = 1 \qquad \text{— (9)}$$

From (7), (8), (9), we infer that

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c} \; ; \; \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c} \; ; \; \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}$$

are direction cosinis of three straight lines, by virtue of (3), (4), (6) are inutually perpendicular.

Hence from the properties of the direction cosines of three mutually perpendicular linis, we have the following six relations:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{a^2} = 1$$

i.e. $x_1^2 + x_2^2 + x_3^2 = a^2$

i.e. $\sum \mathrm{x}_1^2 = a^2$ — (10)

$$\frac{y_1^2}{a^2} + \frac{y_2^2}{a^2} + \frac{y_3^2}{a^2} = 1$$

i.e. $y_1^2 + y_2^2 + y_3^2 = a^2 \Rightarrow \sum y_1^2 = b^2$ — (11)

$$\frac{z_1^2}{c^2} + \frac{z_2^2}{c^2} + \frac{z_3^2}{c^2} = 1$$

$$z_1^2 + z_2^2 + z_3^2 = c^2$$

i.e. $\sum z_1^2 = c^2$ — (12)

and $\frac{x_1 \cdot}{a} \frac{y_1}{b} + \frac{x_2 \cdot}{a} \frac{y_2}{b} + \frac{x_3 \cdot}{a} \frac{y_3}{b} = 0$

$$x_1 y_1 \neq x_2 y_2 + x_3 y_3 = 0$$

i.e. $\sum x_1 y_1 = 0$ — (13)

$$\frac{y_1 \cdot}{b} \frac{z_1}{c} + \frac{y_2 \cdot}{b} + \frac{y_3 \cdot}{b} \frac{z_3}{c} = 0$$

$$y_1 z_1 + y_2 z_2 + y_3 z_3 - 0$$

i.e. $\sum y_1 z_1 = 0$ — (14)

$$\frac{z_1}{c}\cdot\frac{x_1}{a}+\frac{z_2}{c}\cdot\frac{x_2}{a}+\frac{z_3}{c}\cdot\frac{x_3}{a}=0$$

$$z_1x_1+z_2x_2+z_3x_3=0$$

i.e. $\sum z_1 x_1 = 0$ — (15)

Problem (7):- Find the condition that the plane $\ell x+my+nz=p$ may touch the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

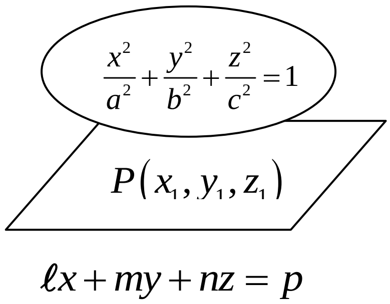
Solution :- The given eq^n. of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ — (1)

Let the plane $\ell x+my+nz=p$ — (2)

touch the ellipsoid at the point $P(x_1, y_1, z_1)$

then, the plane (2) must be identical with $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1$ — (3)

Hence comparing the coefficients of (2) & (3), we get



$$\frac{\frac{x_1}{a^2}}{\ell}=\frac{\frac{y_1}{b^2}}{m}=\frac{\frac{z_1}{c^2}}{n}=\frac{1}{p}$$

$$x_1=\frac{a^2\ell}{p}, y_1=\frac{b^2m}{p}, z_1=\frac{c^2n}{p}$$

Also, since $P(x_1, y_1, z_1)$ is on the ellipsoid (1)

Substituting the values of x_1, y_1 and z_1, we get $\frac{x_1^2}{a^2}+\frac{b^4m^2}{b^2p^2}+\frac{c^4n^2}{c^2p^2}=1$

Or $a^2\ell^2+b^2m^2+c^2n^2=p^2$

This is the required condition.

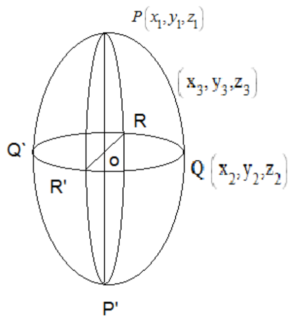
Problem (8):- Define conjugate diameters of an ellipsoid. Prove that the sum of square of any three conjugate semi-diameters of an ellipsoid is a constant.

Definition:- Conjugate diameters of an ellipsoid:- Any chord that passes through the centre of an ellipse is called its diameter. It follows that the family of parallel chords define two diameters: one in the direction to which they are all parallel and the other locus of their mid points. Such two diameters are called conjugate.

Sol^n.:- The eq^n. of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b}+\frac{z^2}{c^2}=1 \qquad \text{— (1)}$$

Its centre is the origin O (o,o,o)



Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid. Then the diametral plane of OP, that is the plane bisecting chord parallel to OP is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \qquad \text{—(2)}$$

Let $Q(x_2, y_2, z_2)$ be any point on the section of the ellipsoid by the diametral plane of OP.

Then, $Q(x_2, y_2, z_2)$ must satisfy (2)

$$\frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2}=0 \qquad \text{— (3)}$$

This shows that the point $P(x_1, y_1, z_1)$ lies on the diametral plane

$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}=$ 0 of OQ.

Thus, if the diametral of OP passes through Q than the diametral plane of OQ also passes through P.

Let OR be the line of intersection of the diametral planes of OP and OQ, where the point $R(x_3, y_3, z_3)$ is on the ellipsoid.

Since, the diametral plane (2) of OP passes through $R(x_3, y_3, z_3)$.

$$\therefore \quad \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0 \qquad \text{— (4)}$$

This shows that the diametral plane $\frac{xx_3}{a^2}+\frac{yy_3+}{b^2}\frac{zz_3}{c^2}=0$ — (5)

of OR must pass through $P(x_1, y_1, z_1)$.

Similarly, the diametral plane (5) of OR must also pass through $Q(x_2, y_2, z_2)$.

$$\therefore \quad \frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0 \qquad \text{— (6)}$$

Hence, the diametral plane of Or is the plane POQ.

We find that the three semi-diameters OP, OQ, OR are such that the diametral plane of any one of them contains the other two.

Hence OP, OQ, OR are called the conjugate semi-diameters of the ellipsoid.

Since, the pts $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ and R(x_3, y_3, z_3) and on the ellipsoid (1)

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1 \qquad \text{— (7)}$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1 \qquad \text{— (8)}$$

$$\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1 \qquad \text{— (9)}$$

From (7), (8), (9), we inter that $\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}; \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}; \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}$ are the direction ratios of three straight line which are mutually $\perp$r by virtue of (5), (4), (6)

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{x_2^2}{a^2}+\frac{x_3^2}{a^2}=1 \Rightarrow x_1^2+x_2^2+x_3^2=a^2 \Rightarrow \sum x_1^2=a^2 \qquad \text{— (10)}$$

$$\frac{y_1^2}{b^2}+\frac{y_2^2}{b^2}+\frac{y_3^2}{c^2}=1 \Rightarrow y_1^2+y_2^2+y_3^2=a^2 \Rightarrow \sum y_1^2=a^2 \qquad \text{— (11)}$$

$$\frac{z_1^2}{c^2}+\frac{z_2^2}{c^2}+\frac{z_3^2}{c^2}=1 \Rightarrow z_1^2+z_2^2+z_3^2=c^2 \Rightarrow \sum z_1^2=a^2 \qquad \text{— (12)}$$

The sum of the squares of three conjugate semi-diameters $= OP^2 + OQ^2 + OR^2$

$$= \left(x_1^2 + y_1^2 + z_1^2\right) + \left(x_2^2 + y_2^2 + z_2^2\right) + \left(x_3^2 + y_3^2 + z_3^2\right)$$

$$= \left(x_1^2 + x_2^2 + x_3^2\right) + \left(y_1^2 + y_2^2 + y_3^2\right) + \left(z_1^2 + z_2^2 + z_3^2\right)$$

$$= a^2 + b^2 + c^2 \qquad \left[\text{from (10, (11) \& (12)}\right]$$

= a constant

****