E-content- B.Sc. (Mathematics), Degree-1

Subject- Mathematics, Paper-II

Topic-5 Analytical Geometry (3-D)

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STRAIGHT LINES, PLANES, COPLANER, DIRECTION CASINES, SHORTEST DISTANCE

Condition for the homogenous second degree equation $ax^2 + by^2 + cz^2 + 2gzx + 2hxy + 2fyz = 0$ to represent a pair of planes.

Solution:- The most general homogeneous form of the equation of the second degree in x, y, z is $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ — (1)

Since the eq^n. (1) is of 2nd degree homogenous in x, y, z the equations of the plane represented by (1) can be taken as

$$\ell_1 x + m_1 y + n_1 z = 0 \quad \text{— (2)}$$

and $$\ell_2 x + m_2 y + n_2 z = 0 \quad \text{— (3)}$$

$\therefore$ (1) will represent a pair of planes if it is identically equivalent to

$$(\ell_1 x + m_1 y + n_1 z)(\ell_2 x + m_2 y + n_2 z) = 0 \quad \text{— (4)}$$

By equating coefficients in (1) & (4) we get

$$\ell_1\ell_2 = a, \quad m_1 m_2 = b, \quad n_1 n_2 = C$$

$$m_1 n_2 + m_2 n_1 = 2f, \quad n_1\ell_2 + n_2\ell = 2g,$$

$$\ell_1 m_2 + m_1 \ell_2 = 2h.$$

By eliminating $\ell_1, m_1, n_1;\ \ell_2, m_2, n_2$ from the above six relations, we get the required condition.

Now, consider the product of two null determinates.

$$\begin{vmatrix} \ell_1 & \ell_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} \ell_2 & \ell_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0$$

or $$\begin{vmatrix} 2\ell_1\ell_2 & \ell_1 m_2 + \ell_2 m_1 & \ell_1 n_2 + \ell_2 n_1 \\ \ell_2 m_1 + \ell_1 m_2 & 2m_1 m_2 & m_1 n_2 + m_2 n_1 \\ n_1\ell_2 + n_2\ell_1 & n_1 m_2 + n_2 m_1 & .\ 2n_1 n_2 \end{vmatrix} = 0$$

or $\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$

or $8\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

This is the required condition in the determinant form.

Expanding this determinant, we get

$$a \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \begin{vmatrix} h & f \\ g & c \end{vmatrix} + g \begin{vmatrix} h & b \\ g & f \end{vmatrix} = 0$$

or $a\left(bc - f^2\right) - h\left(ch - gf\right) + g\left(hf - bg\right) = 0$

or $abc - af^2 - ch^2 + fgh + fgh - bg^2 = 0$

or $abc + \alpha fgh - af^2 - bg^2 - ch^2 = 0$

This is also the required condition.

Problem (1):- To find the condition that the two given straight. lines $\frac{x-x_1}{\ell_1} - \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$, $\frac{x_1-x_2}{\ell_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplaner.

Solution:- The given eqns. Of straight lines are

$$\frac{x-x_1}{\ell_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \text{— (1)}$$

and $$\frac{x-x_2}{\ell_3} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad \text{— (2)}$$

The equation of any plane passing through the first line is

$$a\left(x-x_1\right) + b\left(y-y_1\right) + c\left(z-z_1\right) = 0 \quad \text{— (3)}$$

Where $a\ell + bm + cn = 0$ — (4)

The plane (3) will contain the line (2) if the point (x_2, y_2, z_2) lies on it and the line ⊥r to the normal to it.

The condition for this is

$$a(x_1 - x_2) + b(y_1 - y_2) + (z_1 - z_2) = 0 \qquad \text{— (5)}$$

and $a\ell_1 + bm_1 + cn_1 = 0$ — (6)

Therefore, eliminating a,b,c between (4) (5) & (6) we get.

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ \ell & m & n \\ \ell_1 & m_1 & n_1 \end{vmatrix} = 0 \qquad \text{— (7)}$$

Which is the required condition.

Again eliminating a,b,c between (3), (4) & (5), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \ell & m & n \\ \ell_1 & m_1 & n_1 \end{vmatrix} = 0 \qquad \text{— (8)}$$

Which is the eq^n. of the plane containing the two lines.

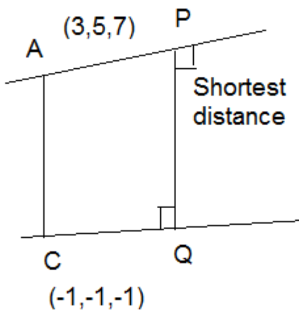
Problem (2):- Define skew lines and find the length and equation of line of shortest distance between lines $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}, \frac{x+1}{-7} = \frac{y+1}{-6} = \frac{z+1}{1}$

Solution:- Let the equation of AB and CD be

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

and

$$\frac{x-1}{-1} = \frac{y-3}{3} = \frac{z-1}{2}$$



The A and C are respectively (3,5,7) and (-1,-1,1). Join A and C. Let (ℓ, m, n) be the direction cosines of the shortest distance PQ between AB and CD.

Then, by the definition of the shortest distance, PQ is ⊥r to both AB and CD.

$$\therefore \quad \ell - 2m + n = 0$$

And $$-7\ell - 6m + n = 0$$

By cross multiplication, we have

$$\frac{\ell}{-2+6} = \frac{M}{-(-1+7)} = \frac{n}{-6-14}$$

$$\frac{\ell}{4} = \frac{M}{-6} = \frac{n}{-20}$$

or $$\frac{\ell}{2} = \frac{M}{-3} = \frac{n}{-10}$$

$$\Rightarrow \frac{\sqrt{\ell^2 + m^2 + n^2}}{\sqrt{4+9+100}} = \frac{1}{\sqrt{113}}$$

$$\therefore \quad \ell = \frac{2}{\sqrt{113}}, M = \frac{-3}{\sqrt{113},}, n = \frac{-10}{\sqrt{113}}$$

(i) The projection of AC on the line PQ is equal to PQ.

$$\therefore \text{ PQ} = (x_2 - x_1)\ell + (y_2 - y_1)m + (z_2 - z_1)n$$

$$= (3+1)\frac{.2}{\sqrt{113}} + (5+1)\frac{-3}{\sqrt{113}} + (7+1)\frac{-10}{\sqrt{113}}$$

$$= \frac{8}{\sqrt{113}} - \frac{18}{\sqrt{113}} - \frac{70}{\sqrt{113}}$$

$$= \frac{8-18-70}{\sqrt{113}}$$

$$= \frac{80}{\sqrt{113}}$$

(ii) The eq^n. of the plane containing AB and PQ is

$$\begin{vmatrix} x\text{-}3 & y-5 & z-7 \\ 1 & -2 & 1 \\ 2 & -3 & -10 \end{vmatrix} = 0$$

or $$(x-3)23 - (y-5) - 12 + (z-7) - 1 = 0$$

$$\Rightarrow 23x - 69 + 12y - 60 - z + 7 = 0$$

$$\Rightarrow 23x + 12y - z - 122 = 0$$

The equation of the plane containing CD and PQ is

$$\begin{vmatrix} x+1 & y+1 & z+1 \\ -7 & -6 & 1 \\ 2 & -3 & -10 \end{vmatrix} = 0$$

or $(x+1)63-(y+1)68+(z+1)33=0$
or $63x+63-68y-68+33z+33=0$
or $63x-68y+33z+28=0$

$\therefore$ The equation of the shortest distance are given by

$$23x+12y-z-122=0 \text{ and } 63x-68y+33z+28=0$$

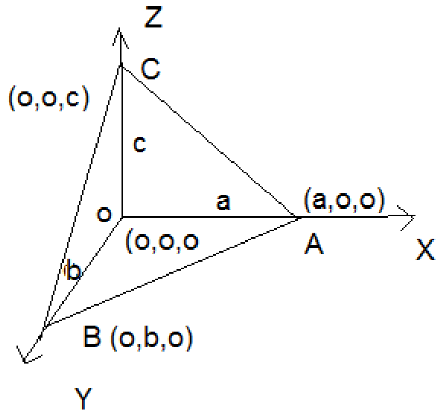
Definition:- Lines which are not parallel and do not intersect are called skew lines.

Problem (3):- Find the equation of plane which makes intercepts a, b & c on the co-ordinate axes respectively.

Solution:- Let the eqn. of the plane is $\ell x+my+nz=p$ — (1)

Where ℓ, m, n are the constants.

Let the plane met the co-ordinate axes in the point A,B, and C respect.



Let OA = a
OB = b
OC = C

Then the co-ordinates of the point A are (a,o,o) and those of B and C are (o,b,o) and (o,o,c) respectively.

Since A lies on the plane, therefore on substituting the co-ordinates of A in (1), we get

$$\ell a = p$$

$$\therefore \ \ell = \frac{p}{a}$$

Similarly, $m=\frac{p}{b}$ and $n=\frac{p}{c}$

Substituting the values of ℓ, m, n in the equations (1), we get

$$\frac{px}{a}+\frac{p}{b}y+\frac{p}{c}z=p$$

i.e. $$\frac{x}{a}+\frac{y}{b}=\frac{z}{c}=p$$

Which is the required equation of the plane.

Problem:- A variable plane meets the coordinate axes at A,B, C such that the centroid of the ΔABC is the point (a,b,c). Show that the equations of the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$

Solution:- Let the coordinates of the point A,B,C be respectively $(\alpha,\text{o},\text{o}),(\text{o},\beta,\text{o}),(\text{o},\text{o},k)$. Therefore, the intercepts of the plane with the co-ordinate axes are (α,β,γ)

Hence, the eq^n. of the plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \qquad \text{— (1)}$$

Since (*a*,b,c) is the centroid of the ΔABC

$$\therefore \quad a=\frac{\alpha}{3} \quad , \quad b=\frac{\beta}{3} \quad , \quad c=\frac{\gamma}{3}$$

i.e. $\alpha=3a \quad , \quad \beta 3b \quad , \quad \gamma 3c$

Substituting the values of in (1), we get

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$$

Which is the required eq^n. of the plane.

Problem (4):- Show that the lines whose direction cosines are given by $\ell+m+n=0$ and $2mn+3nl-6lm=0$ are $\perp r$ to each other.

Solution:- we have

$$\ell+m+n=0$$

Or $\ell = -(m+n)$ — (1)

Again $2mn + 3n\ell - 5\ell m = 0$

or $2mn - 3n(m+n) + 5m(m+n) = 0$ [from (1)]

or $2mn + 3mn - 3n^2 + 5m^2 + 5mn = 0$

or $5m^2 + 4mn - 3n^2 = 0$

$\Rightarrow$ $\frac{5m^2}{n^2} + \frac{4m}{n} - 3 = 0$

This is a quadratic eq[n]. in $\frac{m}{n}$. So it must have two values of $\frac{m}{n}$

Let the two values of $\frac{m}{n}$ be $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$

Where $(\ell, m, n,)$ and (ℓ_2, m_2, n_2) are the direction cosines of the given lines.

Then $\frac{m_1}{n_1} . \frac{m_2}{n_2} = \frac{-3}{5}$

$\therefore \frac{m_1 m_2}{3} = \frac{n_1 n_2}{-5}$ — (2)

From (1) $m = -(\ell + m)$

Now, $2mn + 3n\ell - 5\ell m = 0$

or $-\alpha n(\ell + n) + 3n\ell + 5\ell(\ell + n) = 0$

or $5\ell^2 + 6\ell n - 2n^2 = 0$

$\Rightarrow$ $5\frac{\ell^2}{n^2} + 6\frac{\ell}{n} - 2 = 0$

The rooks of this eq[n]. are $\frac{\ell_1}{n_1}$ $\frac{\ell_2}{n_2}$.

Then $\frac{\ell_1}{n_1} . \frac{\ell_2}{n_2} = -\frac{2}{5}$ or $\ell \frac{\ell_2}{2} = \frac{n_1 n_2}{-5}$ — (3)

From (2) & (3), we get

$$\frac{\ell_1\ell_2}{\alpha} = \frac{m_1m_2}{3} = \frac{n_1n_2}{-5}$$

$$\Rightarrow \quad \frac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{\alpha + 3 - 5}$$

$$\Rightarrow \quad \ell_1\ell_2 + m_1m_2 + n_1n = 0$$

Hence, the given lines we $\perp r$ to each other.

Problem (5):- Find the equation of the plane through the point (1,1,1) and $\perp r$ to the planes $x - 2y + z = 2$ and $4x - 3y - z + 1 = 0$.

Solution:- Let the eq^n. of the required plane be $ax + by + cz + d = 0$

— (1)

Since it is passing through the point $R(1,1,1)$, we get

$$a.1 + b.1 + c.1 + d = 0$$

$$\Rightarrow \quad a + b + c + d = 0 \qquad \text{— (2)}$$

Again, since the plane (1) is $\perp r$ to the given plane $x - 2y + z = 2$ and $4x + 3y - z + 1 = 0$

$$\therefore \quad aa_1 + bb_1 + cc_1 = 0 \qquad \text{(formula)}$$

$$\Rightarrow \quad a.1 + b.(-2) + c.1 = 0$$

$$\Rightarrow \quad a.4 + b.3 + c.(-1) = 0$$

Solving these two eq^n.s by cross multiplication, we get

$$\frac{a}{\alpha - 3} = \frac{b}{4 + 1} = \frac{c}{4 + 8}$$

$$\frac{a}{1} = \frac{b}{5} = \frac{c}{11} = k \text{ (say) where } k \neq 0$$

$$\therefore \quad a = -k, \quad b = 5k, \quad c = 11k.$$

Substituting these values in (2), we get

$$-k + 5k + 11k + d = 0$$

or d = - 15 k.

Putting the values of a,b,c & d in (1), we get

$$-kx + 5ky + 11kz - 15k = 0$$

$$\Rightarrow x-5y-11z+15=0$$

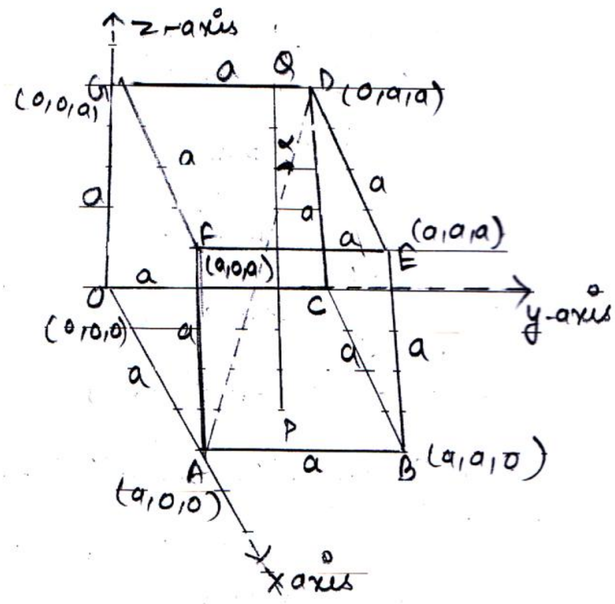
This is the required eq^n. of the plane.

Problem (6):- A line makes an angle $\alpha,\beta,\gamma,\delta$ with the diagonal of a cube. Prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=4/3$

Solution:- Let OABCDEFG be a cube whose each side be a.

Let us take 0 as the origin and OA, OC, OG as x, y, z-axis respectively.

Then, the coordinates of O, A, B, C, D, E, F, G are respectively (o,o,o), (a,o,o), (a,a,o), (o,a,o) (o,a,a), (a,a,a), (a,o,a), (o,o,a).



The diagonal of the cube are AD, BG, CF and OE.

The direction ratios of $AD = x_2-x_1, y_2-y_1, z_1-z_1$

i.e. $\quad o-a,\ a-o,\ a-o$

i.e. $\quad -a,\ \ a,\ \ a$

The direction cosines of AD are

$$\frac{-a}{\sqrt{a^2+a^2+a^2}},\ \frac{a}{\sqrt{a^2+a^2+a^2}},\ \frac{a}{\sqrt{\alpha^2+a^2+a^2}}$$

i.e. $\dfrac{-\not{a}}{\sqrt{3\not{a}^2}}, \dfrac{\not{a}}{\sqrt{3\not{a}^2}}, \dfrac{\not{a}}{\sqrt{3\not{a}^2}}$

$= \dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

Let PQ be any line with direction cosines ℓ, m, n which makes angles $\alpha, \beta, \gamma, \delta$ with the diagonal of the cube.

Then $\cos\alpha = \ell_1\ell_2 + m_1m_2 + mn_2$

$$= \frac{1}{\sqrt{3}}(-\ell + m + n)$$

$$\cos\beta = \frac{1}{\sqrt{3}}(\ell - m + n)$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(\ell + m - n)$$

$$\cos\delta = \frac{1}{\sqrt{3}}(\ell + m + n)$$

$\therefore$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}\left[(-\ell - m + n)^2 + (\ell - m + n)^2 + (\ell + m - n)^2 + (\ell + m + n)^2\right]$$

$$= \frac{1}{3}\Big[\ell^2 + m^2 + n^2 - 2\not{\ell}m + 2\not{m}n - 2\not{\ell}n + \ell^2 + m^2 + n^2$$
$$2\ell\not{m} - 2\not{m}n + 2\not{\ell}n + \ell^2 + m^2 + n^2 + \not{2}\ell m - \not{2}mn$$
$$-2\not{\ell}n + \ell^2 + m^2 + n^2 + 2\not{\ell}m + 2\not{m}n + 2\not{\ell}n\Big]$$

$$= \frac{1}{3}\left[\left(\ell^2 m^2 + n^2\right) + \left(\ell^2 + m^2 + n^2\right) + \left(\ell^2 + m^2 + n^2\right) + \left(\ell^2 + m^2 + n^2\right)\right]$$

$$= \frac{1}{3}(1+1+1+1) \qquad \left[\because\ \ell^2 + m^2 + n^2 = 1\right]$$

$$= \frac{4}{3}$$

Hence, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \dfrac{4}{3}$

Problem (7):- The direction cosines ℓ, m, n of three lines are connected by the relations $\ell + m + n = 0; \alpha\ell n - \alpha mn =$ find them.

Solution:- From the given relations, we have

$$\ell = -(+m+n) \;\; ; \;\; mn - 2\ell(m+n) = 0$$

Eliminating ℓ, we get

$$mn + 2(m+n)^2 = 0$$
$$\Rightarrow 2m^2 + 5mn + 2n^2 = 0$$
$$\Rightarrow (m+2n)(2m+n) = 0$$
$$\Rightarrow m+2n = 0 \quad \text{....(1) or} \quad 2m+n = 0 \quad \text{....(2)}$$

Let the two lines whose direction cosines satisfy the given relations, have their direction cosines (ℓ_1, m_1, n_1) and (ℓ_2, m_2, n_2)

Then from (1)

$$m_1 + 2n_1 = 0$$
$$\therefore \quad m_1 = -2n1$$

But $\ell_1 + m_1 + n_1 = 0$

$$\therefore \quad \ell_1 = -(m_1 + n_1)$$
$$= -(-2n_1 + n_1)$$
$$\ell_1 = n_1$$

Also from (2)

$$2m_2 + n_2 = 0$$

$$\therefore n_2 = -2m_2$$

But $\ell_2 + m_2 + n_2 = 0$

$$\ell_2 = -(m_2 + n_2)$$
$$= (m_2 - 2m_2)$$
$$= +m_2$$

But we know that

$$\ell_1^2 + m_1^2 + n_1^2 = 1$$

$$\therefore \quad n_1^2 + 4_1^2 + n_1^2 = 1$$

$$\Rightarrow n_1^2 = \frac{1}{6}$$

$$\Rightarrow n_1 = \frac{1}{\sqrt{6}}$$

and $\ell_1 = n_1 = \frac{1}{\sqrt{6}}$

and $m_1 = 2n_1 = \frac{-2}{\sqrt{6}}$

Hence, $(\ell, m_1, n_1) = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

Again, $\ell_2^2 + m_2^2 + n_2^2 = 1$

$$m_2^2 + m_2^2 + 4m_2^2 = 1$$

$$\Rightarrow 6m_2^2 = 1$$

$$\Rightarrow m_2 = \frac{1}{\sqrt{6}}$$

$$\therefore \quad \ell_2 = m_2 = \frac{1}{\sqrt{6}}$$

$$n_2 = -2m_2 = \frac{-2}{\sqrt{6}}$$

Hence; $(\ell_2, m_2, n_2) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

Now if θ be the angle b/w the two lines, then

$$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$$

$$= \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} + \left(\frac{-2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) + \frac{1}{\sqrt{6}}\left(\frac{-2}{\sqrt{6}}\right)$$

$$= \frac{1}{6} - \frac{2}{6} - \frac{2}{6}$$

$$= \frac{1-4}{6} = \frac{-3}{6}$$

$$\cos\theta = \frac{-1}{2}$$

The acute angle between the two lines is given by

$$\cos\theta = +\frac{1}{2}$$

$$\therefore \quad \theta = \frac{\lambda}{3}$$

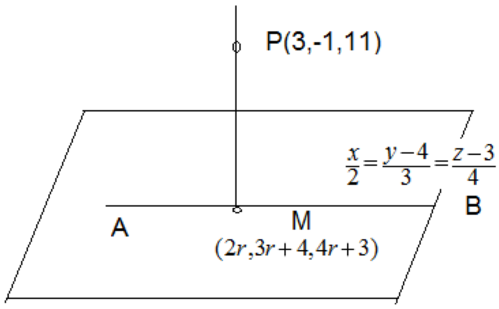
Problem (8):- Find the equation of the line which passes through the point $(3,-1,11)$ and is $\perp r$ to the line $\frac{1}{2}x=\frac{1}{3}(x-y)= \quad \neq(z-3)$

Solution:- The given eqn. of lines is

$$\frac{x}{2}=\frac{y-4}{3}=\frac{z-3}{4}=r \text{ (say)}$$

$\therefore \quad x=2r, y=3r+4, z=4r+3.$

The coordinates of any point on the line AB are $(2r,3r+4,4r+3)$.



Now, the direction cosines of line PM is

$$(2r-3,3r+5,4r-8)$$

And it is given that the line through the point (3,-1,11) is $\perp r$ to the given line

$$\therefore \quad 2(2r-3)+3(3r+5)+4(4r+3)=0$$

$$4r-6+9r+15+16r+12=0$$
$$29r+21=0$$
$$r=\frac{-21}{29}$$

Now, the direction cosins of the line PM are

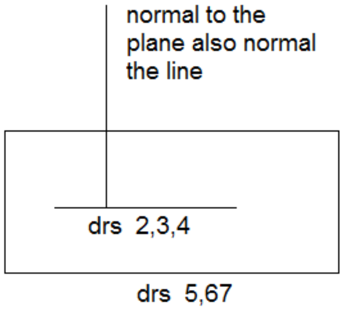
$$\left(2\times\frac{-21}{29}-3,3\times\frac{-21}{29}+5,4\times\frac{-21}{29}-8\right)$$
$$=\left(\frac{-129}{29},\frac{82}{29},\frac{148}{29}\right)$$

$\therefore \qquad$ The required eqn. of the line is

$$\Rightarrow \frac{x-3}{\frac{-129}{219}}, \frac{y+1}{\frac{82}{29}}, \frac{z-11}{\frac{148}{29}}$$

Problem (9):- Find the eqn. of the plane through (-1,0,1) containing the line whose d.c's are proportional to (2,3,4) and (5,6,7).

Solution:- The eqn. of any plane through the point (-1,0,1) is



$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \qquad \text{— (1)}$$
$$\Rightarrow a(x+1)+by+c(z-1)=0$$

Where a,b,c are dr's of the plane.

Whose value is to be found.

If the plane containing the line where d.c's are proportional to the (2,3,4) and (5,6,7)

$\therefore$ Normal to the plane is also normal to both the line

$\therefore$ By formula

$$a_1a_2+b_1b_2+c_1c_2=0$$
$$2a+3b+4c=0$$
$$5a+6b+7c=0$$

By cross multiplication

$$\frac{a}{21-24}=\frac{b}{20-14}=\frac{c}{12-15}$$
$$\Rightarrow \frac{a}{-3}=\frac{b}{6}=\frac{c}{-3}$$
$$\Rightarrow \frac{a}{-1}=\frac{b}{2}=\frac{c}{-1}$$

Multiplying by -1.

We put the value of a,b,c in equation (1) we get the required eqn. of the plane.

$$-1(x+1)+2y-1(z-1)=0$$
$$\Rightarrow -x-\not{1}+2y+z+\not{1}=0$$
$$\Rightarrow -x+2y+z=0$$
$$\Rightarrow x-2y-z=0$$

Problem (10):- Obtain the condition for given two lines

$$\frac{x-\alpha_1}{\ell_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \text{ and } \frac{x-\alpha_2}{\ell_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$$

to be coplanar. Also find the eqn. of the plane containing then in case they are coplanar.

Solution:- Let the two lines are

$$\frac{x-\alpha_1}{\ell_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \qquad \text{— (1)}$$

and

$$\frac{x-\alpha_2}{\ell_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} \qquad \text{— (2)}$$

Let the equation of the plane containing the given lines be

$$ax+by+cz+d=0 \qquad \text{— (3)}$$

The condition are

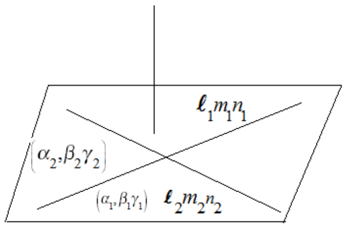
$$a\ell_1+bm_1+cn_1=0 \qquad \text{— (4)}$$

$$a\alpha_1+b\beta_1+c\gamma_1+d=0 \qquad \text{— (5)}$$

and

$$a\ell_2+bm_2+cn_2=0 \qquad \text{— (6)}$$

$$a\alpha_2+b\beta_2+c\gamma_2=0 \qquad \text{— (7)}$$



Subtracting (7) from (8), we get

$$a(\alpha_1-\alpha_2)+b(\beta_1-\beta_2)+c(\gamma_1-\gamma_2)=0 \qquad \text{— (8)}$$

Eliminating a,b,c from (8), (4) & (6), we get,

$$\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=0$$

This is the required condition

Subtracting (5) from (3), we get

$$a(x-\alpha_1)+b(y-\beta_1)+c(z-\gamma_1)=0 \qquad \text{— (9)}$$

Eliminating a,b,c from (9), (4) & (6), we get

$$\begin{vmatrix} x-\alpha_1 & x-\beta_1 & x-\gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=0$$

This is the required eqn. of the plane containing the given lines.

Problem (11):- Show that the eqn. to the plane containing the line $\frac{y}{b}+\frac{z}{c}=1, x=0$ and parallel to $\frac{x}{a}+\frac{-z}{c}=0, y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$. Also prove that if 2d be the shortest distance between three straight line, then $\frac{1}{a^2}-\frac{b}{b^2}-\frac{1}{c^2}=\frac{1}{d^2}$

Solution:- The eqn. of any plane through the first line is

$$\frac{y}{b}+\frac{z}{c}-7+\lambda x=0 \qquad \text{— (1)}$$

The plane (1) will be parallel to the second line if it will be parallel to the plane through the 2^{nd} line.

The equation of any plane through the second line is

$$\frac{x}{a}-\frac{z}{c}-1+\mu y=0 \qquad \text{— (2)}$$

The plane (1) & (2) are parallel if the normals to therm are parallel which is possible only when

$$\frac{\lambda}{1/a} = \frac{1/b}{\mu} = \frac{1/c}{-1/c}$$

$$\therefore \quad \lambda = \frac{-1}{a}, \quad \mu=\frac{-1}{b}$$

Substituting $\lambda=-1/a$ in (1), we get

$$\frac{-x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$$

$$\Rightarrow \quad \frac{x}{a}+\frac{y}{b}-\frac{z}{c}+1=0 \qquad \text{— (3)}$$

Which is the eqn. of the required plane.

The shortest distance between them is the distance of any point on (2) from the plane (3)

$\therefore 2d = \perp r$ from (a,o,o) i.e any pt. on (2) to the plane (3)

$$\not{2}d=\frac{\not{2}}{\sqrt{1/\alpha^2+1/b^2+1/c^2}}$$

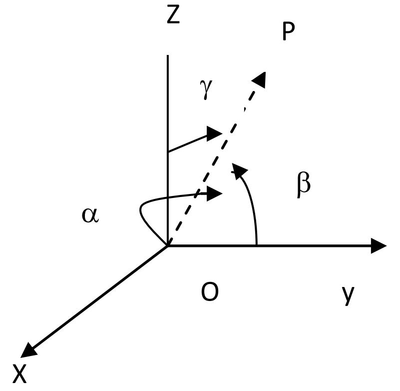
$$d^2 = \frac{1}{1/a^2 + 1/b^2 + 1/c^2}$$

$$\therefore \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{d^2}$$

Problem (12):- Define direction cosines of a straight line. If ℓ, m, n be direction cosines of a straight line, prove that $\ell^2 + m^2 + n^2 = 1$.

Definition:-

When a line in space is taken in a definite sense from one extreme to the other, the line is said to be directed.

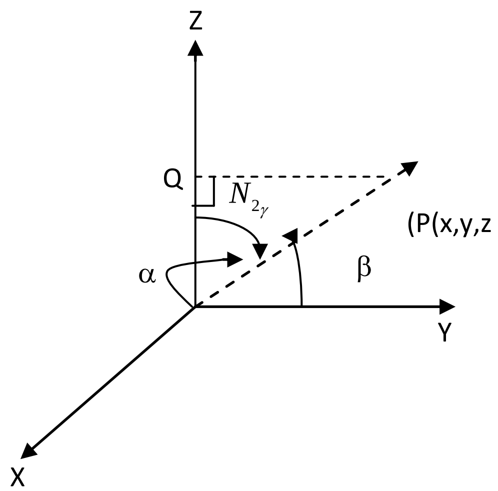


For a directed line OP passing through the origin the angles α, β, γ formed by OP with the $x_1 y_1 z-$ axis respectively are called the direction angles of OP and the cosines of these angles, that is, $\cos\alpha, \cos\beta, \cos\gamma$ are called direction cosines of OP. The direction cosines of a line are generally denoted of a line

by ℓ, m, n.

Then, $\ell = \cos\alpha, m = \cos\beta, n = \cos\gamma$.

Solution:-



Let OP be the directed line through the origin making angles with angles α, β, γ with the axes x, y, z axes respectively.

Consider now a point $P(x, y, z)$ on the directed line.

Let $OP = r$

From P draw $\perp r$ to the Z-axis

Then OQ = Z and $\angle QOP = r$

From right angled ΔQOP

$$\cos\gamma = \frac{OQ}{OP} = \frac{z}{r}$$

$$z = r\cos\gamma.$$

Similarly, drawing $\perp rs$ from P on x- and y-axis we get

$$x = \cos\alpha \text{ and } y = r\cos\beta$$

Now,

$$\frac{OP = r}{\sqrt{x^2+y^2+z^2}} = r$$

$$x^2+y^2+z^2 = r^2$$

$$r^2\cos^2\alpha + r^2\cos^2\beta + r^2\cos^2\gamma = r^2$$

$$r^2\left(\cos^2 + \alpha\cos^2\beta + \cos^2\gamma\right) = r^2$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

If $\ell = \cos\alpha, m = \cos\beta, n = \cos\gamma$

Then $\quad \ell^2 + m^2 + n^2 = 1$

Problem (13):- Transform the Cartesian co-ordinates (3,4,4) of a point to

(i) Spherical polar co-ordinates
(ii) Cylindrical polar co-ordinates

Solution:- (i) Let the cautesian co-ordinates of the point be (x, y, z) and the polar coordinates of the point be (r, θ, ϕ)

Then, $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$

and $z = r\cos\theta$

$$\therefore \quad r^2 = x^2 + y^2 + z^2$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$

$$\tan\phi = \frac{y}{x}$$

Here, $x = 3$, y = 4, z = 5

$$\therefore \quad r^2 = 3^2 + 4^2 + 5^2 = 9 + 16 + 25 = 49$$

$$\therefore r = 7$$

Also, $\tan\theta = \frac{\sqrt{x^2 y^2}}{z} = \frac{\sqrt{3^2 + 4^2}}{}$

$$= \frac{\sqrt{9+16}}{5}$$

$$= \frac{\sqrt{25}}{5}$$

$$= 5/5$$

$$\tan\theta = 1$$

$$\therefore \quad \theta = \overline{\wedge}/4$$

Again, $\tan\phi = y/x = \frac{4}{3}$

$$\therefore \quad \phi = \tan^{-1}\left(4/3\right)$$

Hence, the required spherical polar co-ordinates of the given point are $\left(7, \frac{\lambda}{4}, \tan^{-1}\left(4/3\right)\right)$.

(ii) Let the cartesian co-ordinates of the point be (x, y, z) and the cylindrical polar coordinates of the point (ζ, ϕ, z)

Then $\alpha = \varsigma\cos\phi, y = \varsigma\sin\phi, z = z.$

$$\therefore \quad \varsigma^2 = x^2 + y^2$$

$$\tan\phi = \frac{y}{x}.$$

$$z = z$$

Here $x = 3, y = 4, z = 5$

$$\therefore \quad \varsigma^2 = 3^2 + 4^2 = 9 + 16 = 25$$

$$\varsigma = 5.$$

Also,

$$\tan\phi = y/x = 4/3$$
$$\phi = \tan^{-1} 4/3$$

and $\quad z = 5.$

Hence, the required cylindrical polar coordinates of the given point are

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