Canonical form of Partial Differential Equation

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1 General Second Order P.D.E.

1. **Definition:** A partial differential differential equation is said to be a second order semi-linear p.d.e. if it can be put in the form

$$R(x,y)u_{xx} + S(x,y)u_{xy} + T(x,y)u_{yy} + g(x,y,u_x,u_y) = 0 \tag{1}$$

where $R^2 + S^2 + T^2 \neq 0$ and R, S and T are continuous function of x and y.

2. **Definition:** A function $u = u(x,y)$ is said to be a regular solution of Equation (**??**) in a domain $\mathbf{D} \subseteq \mathbb{R} \times \mathbb{R}$ if $u \in \mathbf{C}^2(\mathbf{D})$, and the function and its derivatives satisfy Equation (**??**) identically in x and y for $(x,y) \in \mathbf{D}$. **Example:** If $f \in \mathbf{c}^2(\mathbb{R})$ then $u = f(x+t)$ is solution of the second order p.d.e.

$$u_{xx} - u_{tt} = 0$$

3. **Analytic function of z :** If $f(z) = u(x,y) + \iota v(x,y)$ is analytic in z $(= x + \iota y)$, then it is well known that u and v satisfy the Cauchy-Riemann equations

$$u_x = v_y \quad , \quad u_y = -v_x$$

Therefore

$$u_{xx} + u_{yy} = 0$$

Similarly

$$v_{xx} + v_{yy} = 0$$

are known Laplace Equation.

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1.1 Classification of Second Order P.D.E. in Two variables

Consider the following second order semi-linear p.d.e.

$$Lu + g(x, y, u_x, u_y) = 0 \tag{2}$$

where

$$L = R(x, y)\frac{\partial^2}{\partial x^2} + S(x, y)\frac{\partial^2}{\partial x \partial y} + T(x, y)\frac{\partial^2}{\partial y^2}$$

, $R^2 + S^2 + T^2 \neq 0$ and R, S and T are continuous function of x and y. we further assumed that this functions posses partial derivatives with respect to x and y. We change the variables (x, y) to (ξ, η).

$$\xi = \xi(x, y) \quad , \quad \eta = \eta(x, y)$$

Note: We assume that $\xi_x\eta_y - \xi_y\eta_x \neq 0$ so that the transformation is invertible at least locally.
it is easy that

$$\begin{aligned}
&u_x = u_\xi\xi_x + u_\eta\eta_x \\
&u_y = u_\xi\xi_y + u_\eta\eta_y \\
&u_{xx} = u_{\xi\xi}\xi_x^2 + u_{\eta\xi}\eta_x\xi_x + u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\eta\eta_{xx} + u_\xi\xi_{xx} \\
&u_{yy} = u_{\xi\xi}\xi_y^2 + u_{\eta\xi}\eta_y\xi_y + u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\eta\eta_{yy} + u_\xi\xi_{yy} \\
&u_{xy} = u_{\xi\xi}\xi_x\eta_y + u_{\eta\xi}\eta_x\xi_y + u_{\xi\eta}\xi_x\eta_y + u_{\eta\eta}\eta_x\eta_y + u_\eta\eta_{xy} + u_\xi\xi_{xy}
\end{aligned}$$

Therefore

$$\begin{aligned}
&R(x, y)\tfrac{\partial^2}{\partial x^2} + S(x, y)\tfrac{\partial^2}{\partial x \partial y} + T(x, y)\tfrac{\partial^2}{\partial y^2} = \\
&u_{\xi\xi}\Big(R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2\Big) + u_{\xi\eta}\Big[2R\xi_x\eta_x + S\Big(\xi_x\eta_y + \xi_y\eta_x\Big) + 2T\xi_y\eta_y\Big] \\
&+u_{\eta\eta}\Big(R\eta_x^2 + S\eta_x\eta_y + T\eta_y^2\Big)
\end{aligned}$$

Therefore Equation (**??**) becomes.

$$A(\xi_x, \xi_y)u_{\xi\xi} + 2B(\xi_x, \xi_y; \eta_x, \eta_y)u_{\xi\eta} + A(\eta_x, \eta_y)u_{\eta\eta} = G(\xi, \eta, u; u_\xi, u_\eta) \tag{3}$$

where

$$A(u, v) = Ru^2 + Suv + Tv^2 \tag{4}$$

$$B(u_1, v_1; u_2, v_2) = Ru_1u_2 + S(u_1v_2 + u_2v_1) + Tv_1v_2 \tag{5}$$

Then we show that

$$A(\xi_x, \xi_y)A(\eta_x, \eta_y) - B^2(\xi_x, \xi_y; \eta_x, \eta_y) = \Big(4RT - S^2\Big)\Big(\xi_x\eta_y - \xi_y\eta_x\Big)/4 \tag{6}$$

Choose ξ and η takes simple form.

Case i. if $S^2 - 4RT > 0$, then p.d.e. is Hyperbolic type. For this we chosen ξ and η that coefficient of $u_{\xi\xi}$ and $u_{\eta\eta}$ in equation (**??**) vanish.
Consider $R\alpha^2 + S\alpha + T = 0$, This equation has two real distinct roots $\lambda_1(x, y)$ and $\lambda_2(x, y)$.
We choose ξ and η such that

$$\frac{\partial \xi}{\partial x} = \lambda_1\frac{\partial \xi}{\partial y}, \qquad \frac{\partial \eta}{\partial x} = \lambda_2\frac{\partial \eta}{\partial y}$$

This is first order p.d.e. for ξ and η. if $f_1(x, y) = c_1$ and $f_2(x, y) = c_2$ are solution of ordinary differential equation.

$$\frac{dy}{dx} + \lambda_1(x, y) = 0, \qquad \frac{dy}{dx} + \lambda_2(x, y) = 0$$

respectively, then

$$\xi = f_1(x, y) \qquad \eta = f_2(x, y)$$

For this choice of ξ and η

$$A(\xi_x, \xi_y) = A(\eta_x, \eta_y) = 0$$

From equation (**??**), in this condition ,we have $B^2 > 0$. Hence equation (**??**) reduce to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \phi(\xi, \eta, u, u_\xi, u_\eta) \tag{7}$$

Eg: $u_{tt} = c^2 u_{xx}$ is one dimensional wave equation. which is hyperbolic.

Case ii. If $S^2 - 4RT = 0$, then p.d.e. is Parabolic type. In this case roots of $R\alpha^2 + S\alpha + T = 0$ coincide (say $\lambda(x, y)$).

Define $\xi = f(x, y)$, where $f(x, y) = c$ is solution of

$$\frac{dy}{dx} + \lambda(x, y) = 0$$

Take η as arbitrary function of x and y such that $\xi_x \eta_y - \xi_y \eta_x \neq 0$. For this $B = 0$ and $A(\xi_x, \xi_y) = 0$.

Since η is independent of ξ, $A(\eta_x, \eta_y) \neq$. Then equation (**??**) becomes

$$\frac{\partial^2 u}{\partial \xi^2} = \phi(\xi, \eta, u, u_\xi, u_\eta) \tag{8}$$

Eg: $\frac{\partial u}{\partial t} = k u_{xx}$ is Heat equation. which is parabolic.

Case iii. $S^2 - 4RT < 0$

In **Case** 1. roots are complex conjugate. Assume

$$\alpha = \frac{1}{2}(\xi - \eta) \qquad \beta = \frac{\iota}{2}(\eta - \xi)$$

then equation (**??**) becomes

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = \phi(\alpha, \beta, u, u_\alpha, u_\beta) \tag{9}$$

is Elliptic.

Eg: $u_{xx} + u_{yy} = 0$ Laplace equation.

Canonical: Equations (**??**), (**??**) and (**??**) are said to be canonical forms of hyperbolic, parabolic and elliptic type of second order partial differential equation (**??**).

Example:1. Reduce the equation $u_{xx} - x^2 u_{yy} = 0$ to a Canonical form.
Solution: In this case $R = 1, \quad S = 0, \quad T = -x^2$.
Then

$$\begin{aligned}
&S^2 - 4RT = 4x^2 > 0, \quad \forall x \neq 0, \quad \text{hyperbolic type} \\
&\text{Now } \; R\alpha^2 + S\alpha + T = 0, \quad \Rightarrow \quad \alpha^2 - x^2 = 0 \\
&\therefore \quad \alpha = x, \; \text{ and } \; \alpha = -x \\
&\Rightarrow \lambda_1 = x, \quad \lambda_2 = -x \\
&\text{then, } \; \frac{dy}{dx} + x = 0, \quad \frac{dy}{dx} - x = 0 \\
&\Rightarrow \quad y + \frac{x^2}{2} = c_1 \quad \text{and} \quad y - \frac{x^2}{2} = c_2
\end{aligned}$$

$$\text{Therefore} \quad \xi = y + \frac{x^2}{2} \quad \text{and} \quad \eta = y - \frac{x^2}{2} \tag{10}$$

Differentiating with respect to x and y of equation (**??**)

$$\xi_x = x \qquad \xi_y = 1 \;\text{ and }\; \eta_x = -x \qquad \eta_y = 1$$

$$\begin{aligned}
&u_x = u_\xi x - u_\eta y \qquad u_y = u_\xi + u_\eta \\
&u_{xx} = x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} + u_\xi - u_\eta \\
&u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\
&\therefore \quad u_{xx} - x^2 u_{yy} \; = 0 \\
&\Rightarrow \qquad u_{\eta\eta} = \frac{1}{4x^2}(u_\xi - \eta) = \frac{1}{4(\xi - \eta)}(u_\xi - \eta)
\end{aligned}$$

which is required solution.

1.2 Classification of Second Order P.D.E. in n- variables

Consider slandered second order semi-linear p.d.e. in the case of n-variables

$$\Sigma_{i=1}^n \Sigma_{j=1}^n a_{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} = 0 \tag{11}$$

where $a_{ij}(x_1, x_2, x_3, ..., x_n), \quad (\forall i, \quad j = 1, 2, 3, ..., n)$
From concept of linear algebra equation (**??**) form a quadratic form. Character of (**??**) characterized by characteristic value (Eigenvalue).

1. All eigenvalue of p.d.e. (**??**) has same sign then p.d.e.(**??**) will be **Elliptic** type.
2. AT least one pair of eigenvalue of p.d.e. (**??**) has opposite sign then p.d.e.(**??**) will be **Hyperbolic** type.
3. At least one eigenvalue of p.d.e. (**??**) is zero then p.d.e.(**??**) will be **parbolic** type .

Example:2. $u_{xx} + u_{yy} + u_{zz} = 0$ is elliptic type p.d.e. Because this p.d.e. has associate matrix of order 3×3 and

$$\begin{aligned}
&a_{ij} = 0 \qquad i \neq j \quad i, \quad j = 1, 2, 3. \\
&a_{ij} = 1 \qquad i = j \quad i, \quad j = 1, 2, 3.
\end{aligned}$$

Therefore, all eigenvalue is 1, which is positive. Hence given p.d.e. is elliptic type.

Example:3. $u_t - k\Big(u_{xx} + u_{yy} + u_{zz}\Big) = 0$ is parabolic type p.d.e. Because this p.d.e. has associate matrix of order 4×4 and

$$a_{ij} = 0 \qquad i \neq j \quad i, \quad j = 1, 2, 3, 4.$$
$$a_{ij} = -1 \qquad i = j \quad i, \quad j = 1, 2, 3.$$
$$a_{44} = 0$$

Therefore, three eigenvalue is -1 and one eigenvalue is 0(zero). Hence given p.d.e. is parabolic type.

Example:4. $u_{tt} - c^2\Big(u_{xx} + u_{yy} + u_{zz}\Big) = 0$ is hyperbolic type p.d.e. Because this p.d.e. has associate matrix of order 4×4 and

$$a_{ij} = 0 \qquad i \neq j \quad i, \quad j = 1, 2, 3, 4.$$
$$a_{ij} = -1 \qquad i = j \quad i, \quad j = 1, 2, 3.$$
$$a_{44} = 1$$

Therefore, three eigenvalue is -1 and one eigenvalue is 1. one pair eigenvalue is opposite sign. Hence given p.d.e. is hyperbolic type.

Example:5. Determine the region where the following equation is hyperbolic, elliptic or parabolic type.

$$u_{xx} - 2x^2 u_{xz} + u_{yy} + u_{zz} = 0.$$

Ans. : Hyperbolic if $\Big|x\Big| > 1$, parabolic if $\Big|x\Big| = 1$ and elliptic if $\Big|x\Big| < 1$.

...............All the best.........