Concept of wave and Heat Equation

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1 Wave Equation

We know that the most important homogeneous hyperbolic p.d.e. is the wave equation, which can be written as follows

$$u_{tt} = c^2 u_{xx} \tag{1}$$

where C is the wave speed.

1. **One Dimensional Wave Equation**

   - **Transverse vibration of string:** If the string of uniform linear mass density is stretched to uniform tension T and if in equilibrium position, the string coincide with x-axis, then when the string is slightly disturbed from its equilibrium position, the transverse displacement $u(x, t)$ satisfy one Dimensional wave equation

   $$u_{xx} = \frac{1}{c^2} u_{tt}$$

   - **Longitudinal Vibration in a bar:** If a uniform bar bar of elastic material of uniform cross section whose axis is coincide with x-axis is stressed in such a way that each point of a typical cross-section of bar table the same displacement $u(x, t)$, then

   $$u_{xx} = \frac{1}{c^2} u_{tt}$$

   where $c^2 = \frac{E}{\rho}$, E is young's modulus and ρ is density of material.

2. **Two Dimensional Wave Equation:** If a uniform thin elastic membrane of density ρ is stretchered to a uniform tension T and it coincide with xy-axis in the equilibrium position then two dimensional wave equation

   $$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$$

   where $c^2 = \frac{T}{\rho}$.

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3. **Three Dimensional Wave Equation:** if the small disturbance propagation in compressible medium and resulting moyion is irrational. It is governed by thee dimensional wave equation is given by

$$u_{xx} + u_{yy} + u_{zz} = \frac{1}{c^2} u_{tt}$$

1.1 Derivation of one dimensional wave equation i.e.,Transverse vibration of string

Let $u = u(x,t)$ be the transverse displacement from the mean position (x-axis) of a string at time t at the pont x. Consider small portion of the string Δs between the two point P and Q. the force acting on this part of the string are tension T at P and Q. We neglect the weight of the string.

The equation of motion are refer to Fig(1)

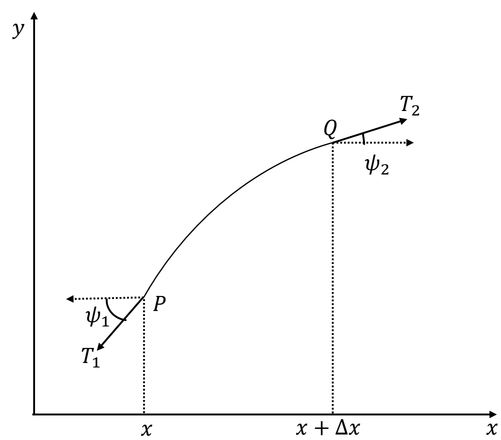

Figure 1:

In x-direction(assuming no displacement in the x-direction)

$$T_2 \cos\psi_2 = T_1 \cos\psi_1$$

in y-direction

$$\rho \Delta s u_{tt} = T_2 \sin\psi_2 - T_1 \sin\psi_1$$

,

$$= T(\tan\psi_2 - \tan\psi_1)$$

, where ρ is the linear density of string.

Observe

$$\tan\psi_1 = (u_x)\Big|_P, \quad \tan\psi_2 = (u_x)\Big|_Q \approx (u_x)\Big|_P + (u_{xx})\Big|_P \Delta x$$

,

Where $\Big|_P$ and $\Big|_Q$ indicates the evaluation at the point P and Q respectively. Hence

$$\rho\Delta s u_{tt} = T\Big((u_x)\Big|_P + (u_{xx})\Big|_P \Delta x - (u_x)\Big|_P\Big)$$

Taking the limit as $\Delta s \to 0$ after dividing both side of the equation by Δx, we obtain

$$\rho u_{tt} = \frac{T u_{xx}}{\sqrt{1+u_t^2}}$$

If $\Big|u_t\Big| << 1$(i.e., if the slope is everywhere small), we get

$$u_{xx} = \frac{1}{c^2} u_{tt} \tag{2}$$

where

$$c^2 = \frac{T}{\rho}$$

Equation (2) is one dimensional wave equation.

2 Heat Conduction Equation

Let us consider a homogeneous,isotropic solid. (Homogeneous means that the material properties are translational invariant and isotropic means that the material properties are same in all direction). Let V be any arbitrary volume inside the solid bounded by surface S. Let δV be a volume element. The heat energy stored in δV is equal to $c\rho u\delta V$, where c is the specific heat of the solid, ρ is density and u is the temperature, which is a function of position and time. Therefore

$$\text{the total heat energy in} V = \iiint\limits_V \mathrm{c}\rho \mathrm{u d}V$$

Let δS be a surface element. The heat flow across $\delta S = \kappa\nabla u \cdot \overline{\mathrm{n}}\delta S$, where $\overline{\mathrm{n}}$ is the outward drawn normal to the surface S and κ is thermal conductivity of the solid. Therefore, the total flux across S, using divergence theorem, is given as follows

$$\iint\limits_S \kappa\nabla u \cdot \overline{\mathrm{n}}\mathrm{d}S = \iiint\limits_V \nabla\cdot(\kappa\nabla u)\mathrm{d}V$$

$$\text{The rate of change of heat energy in} V = \text{the flux of heat energy across} S, (i.e.,)$$

$$\tfrac{d}{dt}\iiint\limits_V \mathrm{c}\rho \mathrm{u d}V = \iiint\limits_V \nabla\cdot(\kappa\nabla u)\mathrm{d}V$$

$$\iiint\limits_V \Big(\tfrac{\partial(c\rho u)}{\partial t} - \nabla\cdot(\kappa\nabla u)\Big)\mathrm{d}V = 0$$

As V is arbitrary and integral is continuous, we have

$$c\rho\tfrac{\partial u}{\partial t} - \nabla\cdot(\kappa\nabla u) = 0$$

if the conductivity κ is constant throughout the body, then

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u \tag{3}$$

which is referred to as the heat conduction equation, where $k = \frac{\kappa}{c\rho}$.

1. In one-dimensional case, the equation (3) becomes

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \tag{4}$$

2. In two-dimensional case, the equation (3) becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\frac{\partial u}{\partial t} \tag{5}$$

3. In three-dimensional case, the equation (3) becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t} \tag{6}$$

4. In three-dimensional case from equation (6), if $x = r\cos\theta,\ \ y = r\sin\theta$ and $z = z$, then co-ordinate system is known cylindrical co-ordinate. The heat equation (6) in cylindrical co-ordinate (r, θ, z) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t} \tag{7}$$

5. In three-dimensional case from (6), if $x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\cos\phi$ and $z = rcos\theta$, then co-ordinate system is known spherical co-ordinate system. The heat equation (6) in spherical co-ordinate (r, θ, ϕ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{k}\frac{\partial u}{\partial t} \tag{8}$$

............**All the best**.........