Solution of wave Equation

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1 D'Alembert Solution of Wave Equation

1.1 Vibration of an Infinite String

Consider the following one dimensional wave equation

$$u_{xx} = \frac{1}{c^2} u_{tt}, \quad \infty < x < \infty, \quad t > 0 \tag{1}$$

due vibration of infinite string.

Solution : Given Equation (1) is hyperbolic p.d.e. because $S^2 - 4RT > 0$, so if we introduce the characteristic curve(variables)

$$\xi = x - ct, \qquad \eta = x + ct$$

Therefore

$$\begin{aligned} u(x,t) &= F(\xi) + G(\eta) \\ &= F(x - ct) + G(x + ct) \end{aligned} \tag{2}$$

This gives the general solution of wave equation (1).

Now we have to find arbitrary function F and G to be $\mathbf{C}^2$ function so that $u(x,t)$is regular solution of equation (1). For this required initial condition.

Let initial condition be given as follows

$$\begin{aligned} u(x,0) &= f(x), \qquad \infty < x < \infty \\ u_t(x,0) &= g(x), \qquad \infty < x < \infty \end{aligned} \tag{3}$$

where $u(x) = f(x)$ is the initial position of the string and $g(x)$ is the initial velocity at the point x. Hence from equations (2) and (3)

$$u(x,0) = f(x) \qquad \Rightarrow \qquad F(x) + G(x) = f(x) \tag{4}$$

and

$$u_t(x,0) = g(x) \qquad \Rightarrow \qquad cF' - cG' = g(x) \tag{5}$$

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then integrating equation (5), we get

$$-F(x)+G(x)=\frac{1}{c}\int_{x_0}^{x} g(s)ds+k \tag{6}$$

Solving equations (4) and (3) ,we get

$$\begin{aligned} F(x) &= \tfrac{1}{2}f(x)-\tfrac{1}{2c}\int_{x_0}^{x} g(s)ds-\tfrac{k}{2}\\ G(x) &= \tfrac{1}{2}f(x)+\tfrac{1}{2c}\int_{x_0}^{x} g(s)ds+\tfrac{k}{2} \end{aligned} \tag{7}$$

we have

$$\begin{aligned} &u(x,t)=F(x-ct)+G(x+ct)\\ &=\tfrac{1}{2}f(x-ct)-\tfrac{1}{2c}\int_{x_0}^{x-ct} g(s)ds-\tfrac{k}{2}+\tfrac{1}{2}f(x+ct)+\tfrac{1}{2c}\int_{x_0}^{x+ct} g(s)ds+\tfrac{k}{2}\\ &\Rightarrow\ u(x,t)=\tfrac{1}{2}\Big[f(x+ct+f(x-ct\Big]+\tfrac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \end{aligned} \tag{8}$$

This is the required solution is known as D'Alembert solution.
If initial velocity $g(x)=0$ then

$$u(x,t)=\frac{1}{2}\Big[f(x+ct+f(x-ct\Big] \tag{9}$$

1.2 Vibration of a Semi-infinite String with fixed end

Let us consider the wave equation is

$$u_{tt}=\tfrac{1}{c^2}u_{xx},\quad 0<x<\infty,\quad t>0 \tag{10}$$

The initial conditions are

$$u(x,0)=f(x),\qquad u_t(x,0)=g(x),\qquad x\geq 0 \tag{11}$$

The boundary condition is

$$u(0,t)=0,\qquad t\geq 0\ \ \text{(i.e., The end x=0 is fixed)} \tag{12}$$

is known fixed end condition **Solution.** The general solution of equation is

$$u(x,t)=\phi(x+ct)+\psi(x-ct) \tag{13}$$

Case :(i) if $x>ct$ then solution of equation (10) same as infinite string case,i.e.,

$$u(x,t)=\tfrac{1}{2}\Big[f(x+ct+f(x-ct\Big]+\tfrac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \tag{14}$$

This solution is meaningless if $x<ct$. For this we consider

Case :(ii) if $x<ct$ then intervals be $[x-ct,x+ct]$, we will go to negative x-axis.
Given,

$$\begin{aligned} u(0,t)=0 \qquad &\Longrightarrow\ \phi(ct)+\psi(-ct)=0\\ &\Longrightarrow\ \psi(-ct)=-\phi(ct) \end{aligned} \tag{15}$$

put $-ct=\alpha$, then

$$\begin{aligned} &\implies \psi(\alpha) = -\phi(-\alpha) \\ &\implies \psi(x) = -\phi(-x) \\ &\implies \psi(x - ct) = -\phi(ct - x) \end{aligned} \tag{16}$$

Then we have according to equation (7)

$$\phi(x) = \tfrac{1}{2}f(x) + \tfrac{1}{2c}\int_0^x g(s)ds + \tfrac{k}{2} \tag{17}$$

From equation (15)

$$\begin{aligned} \phi(ct - x) &= \tfrac{1}{2}f(ct - x) + \tfrac{1}{2c}\int_0^{ct-x} g(s)ds + \tfrac{k}{2} \\ \phi(x + ct) &= \tfrac{1}{2}f(x + ct) + \tfrac{1}{2c}\int_0^{x+ct} g(s)ds + \tfrac{k}{2} \end{aligned} \tag{18}$$

From equations (**??**) and (13)

$$u(x,t) = \phi(x + ct) + \psi(x - ct) = \phi(x + ct) - \phi(ct - x)$$

Hence

$$u(x,t) = \tfrac{1}{2}\Big[f(x + ct) - f(ct - x)\Big] + \tfrac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds, \qquad ct \geq x \tag{19}$$

1.3 Vibration of a Semi-infinite String with free end

Let us consider the wave equation is

$$u_{tt} = \tfrac{1}{c^2}u_{xx}, \quad 0 < x < \infty, \quad t > 0 \tag{20}$$

The initial conditions are

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad x \geq 0 \tag{21}$$

The boundary condition is

$$u_x(0,t) = 0, \qquad t \geq 0 \ \ \text{(i.e., free end x=0 condition)} \tag{22}$$

is known free end condition.
same calculation as fixed end condition

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}\Big[f(x + ct) + f(ct - x)\Big] + \tfrac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds, \qquad x < ct \\ u(x,t) &= \tfrac{1}{2}\Big[f(x + ct) + f(x - ct)\Big] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, \end{aligned} \tag{23}$$

1.4 Some Important Examples:

Question:1. Find $u(\frac{\pi}{4}, \frac{1}{2})$ and $u(\frac{\pi}{2}, \frac{1}{2})$ of

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad c > 0 \\ u(x,0) &= \sin x = f(x) \\ u_t(x,0) &= \cos x = g(x) \end{aligned} \tag{24}$$

Solution:-

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big[f(x+ct) + f(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \\ &= \frac{1}{2}\Big[\sin(x+ct) + \sin(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(s)ds \\ &= \sin x \cos ct + \frac{1}{c}\cos x \sin ct \end{aligned}$$

Hence,

$$u(\frac{\pi}{4}, \frac{1}{2}) = \frac{1}{2\sqrt{2}}(\sin 1 + \cos 1), \quad \textbf{if} \quad c = 2$$

and

$$u(\frac{\pi}{2}, \frac{1}{2}) = \cos 1, \quad \textbf{if} \quad c = 2$$

.............**All the best**.........