Concept of Separation of Variables Method

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Paper: *11* **M.Sc. Part:** II

1 Separation of Variables Method

This method can be used to solve initial as well as boundary value problems.

Let us consider a second order p.d.e. be

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \tag{1}$$

Case I: where A,B,C,D,E and F are not constant but variables,

$$au_{xx} + bu_{yy} + du_x + eu_y + fu = 0 \tag{2}$$

is Canonical form of equation (1).

Let

$$u(x,y) = X(x)Y(y) \neq 0$$

$$\begin{aligned} &aX''Y + bXY'' + eXY' + dYX' + fXY = 0 \\ \Rightarrow \quad &(aX'' + dX' + fX)Y + (bY'' + eY')X = 0 \end{aligned} \tag{3}$$

$\exists$ a function $p(x,y)$ such that when we divide equation (3) by $p(x,y)$, the equation (3) becomes

$$a_1(x)X''Y + b_1(y)XY'' + b_2(y)XY' + a_2(x)YX' + \Big(a_3(x) + b_3(y)\Big)XY = 0$$

$$\Rightarrow \quad \left[\frac{a_1(x)X'' + a_2(x)YX'}{X} + a_3(x)\right] = -\left[\frac{b_1(y)Y'' + b_2(y)Y'}{Y} + b_3(y)\right]$$

Differentiating w.r.t. x , we get

$$\frac{d}{dx}\Big(LHS\Big) = 0$$

$$\Rightarrow \quad \frac{a_1(x)X'' + a_2(x)YX'}{X} + a_3(x) = Constant = \lambda, \qquad \textbf{say}$$

Further

$$\frac{b_1(y)Y'' + b_2(y)Y'}{Y} + b_3(y) = -\lambda$$

λ is known separation constant

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Case II: where A,B,C,D,E and F are constants

Let

$$u(x,y) = X(x)Y(y) \neq 0 \quad \text{be a required solution}$$

$$AX''Y + BX'Y' + CXY'' + DYX' + EXY' + FXY = 0 \tag{4}$$

Equation (4) divided by AXY then

$$\frac{X''}{X} + \frac{B}{A}\frac{X'}{X}\frac{Y'}{Y} + \frac{C}{A}\frac{Y''}{Y} + \frac{D}{A}\frac{X'}{X} + \frac{E}{A}\frac{Y'}{Y} + \frac{F}{A} = 0$$

Differentiating with respect to x

$$\left(\frac{X''}{X}\right)' + \frac{B}{A}\left(\frac{X'}{X}\right)'\frac{Y'}{Y} + \frac{D}{A}\left(\frac{X'}{X}\right)' = 0$$

$$\implies \frac{\left(\frac{X''}{X}\right)'}{\frac{B}{A}\left(\frac{X'}{X}\right)'} + \frac{Y''}{Y} + \frac{D}{A} = 0$$

$$\therefore \quad \frac{\left(\frac{X''}{X}\right)'}{\frac{B}{A}\left(\frac{X'}{X}\right)'} + \frac{D}{A} = -\frac{Y''}{Y} = \lambda$$

$\lambda -$ separation constant.

$$Y' + \lambda Y = 0 \tag{5}$$

which is first orerd differential equation

and

$$\left(\frac{X''}{X}\right)' + \left(\frac{D}{A} - \lambda\frac{B}{A}\right)\left(\frac{X'}{X}\right)' = 0$$

$$\left(\frac{X''}{X}\right)' + \left(\frac{D}{A} - \lambda\frac{B}{A}\right)\left(\frac{X'}{X}\right)' = 0 \tag{6}$$

Integrating with respect to x, we get

$$\frac{X''}{X} + \left(\frac{D}{A} - \lambda\frac{B}{A}\right)\frac{X'}{X} = \beta$$

which is second orerd differential equation

1.1 Solution of one dimensional wave equation by separation of variables method

Consider one dimensional wave equation

$$u_{tt} = \frac{1}{c^2}u_{xx} \tag{7}$$

Suppose the solution of equation (7) is of the form

$$u(x,t) = X(x)T(t) \neq 0 \tag{8}$$

putting the value from equation (8) in equation (6)

$$X''T = \frac{1}{c^2}T''X$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{T''}{c^2T} = \mu \quad \textbf{(say)} \tag{9}$$

where μ is separation constant.

From equation (9), we can deduce that

$$X'' - \mu X = 0 \tag{10}$$

and

$$T'' - c^2\mu T = 0 \tag{11}$$

solving equations (10) and (11), we get

Case I. when $\mu = 0$ then equations (6)and (7)

$$\begin{aligned} &\Rightarrow \quad X(x) = a_1x + a_2, \quad T(t) = a_3t + a_4 \\ &u(x,t) = (a_1x + a_2)(a_3t + a_4) \end{aligned} \tag{12}$$

Case II. when $\mu = -\lambda^2$ $(\lambda \neq 0)$ then equations (10)and (11)

$$\begin{aligned} &\Rightarrow \quad X(x) = b_1e^{\lambda x} + a_2e^{-\lambda x}, \quad T(t) = b_3e^{\lambda ct} + b_4e^{-\lambda ct} \\ &u(x,t) = (b_1e^{\lambda x} + a_2e^{-\lambda x})(b_3e^{\lambda ct} + b_4e^{-\lambda ct}) \end{aligned} \tag{13}$$

Case III. when $\mu = \lambda^2$ $(\lambda \neq 0)$ then equations (10)and (11)

$$\begin{aligned} &\Rightarrow \quad X(x) = d_1\cos\lambda x + d_2\sin\lambda x, \quad T(t) = d_3\cos\lambda ct + d_4\sin\lambda ct \\ &u(x,t) = (d_1\cos\lambda x + d_2\sin\lambda x)(d_3\cos\lambda ct + d_4\sin\lambda ct) \end{aligned} \tag{14}$$

Finally, since we are dealing with problems on vibration, $u(x,t)$ must be periodic function of t. so $u(x,t)$ must be involves trigonometric terms. Hence, the solution equation (7) is

$$u(x,t) = (d_1\cos\lambda x + d_2\sin\lambda x)(d_3\cos\lambda ct + d_4\sin\lambda ct)$$

Only solution of equation (7).

1.2 Solution of one dimensional Heat equation by separation of variables method

Consider one dimensional heat equation

$$u_t = \kappa u_{xx} \tag{15}$$

Suppose the solution of equation (15) is of the form

$$u(x,t) = X(x)T(t) \neq 0 \tag{16}$$

putting the value from equation (16) in equation (15)

$$\begin{aligned} X''T &= \frac{1}{\kappa}T'X \\ \Rightarrow \quad \frac{X''}{X} &= \frac{T'}{\kappa T} = \mu \quad \textbf{(say)} \end{aligned} \tag{17}$$

where μ is separation constant.
From equation (17), we can deduce that

$$X'' - \mu X = 0 \tag{18}$$

and

$$T' - \kappa\mu T = 0 \tag{19}$$

solving equations (18) and (19), we get

Case I. Let $\mu = 0$. Then solution of equations (18) and (19) are given by

$$\begin{aligned} & X = a_1 x + a_2 \quad \text{and} \quad T = a_3 \\ & \therefore \quad u(x,t) = e_1 x + e_2 \\ & \text{where } \; e_1 = a_1 a_3, \quad e_2 = a_2 a_3 \end{aligned} \tag{20}$$

Case II. Let $\mu = \lambda^2 \quad \lambda \neq 0$,Then solution of equations (18) and (19) are given by

$$\begin{aligned} & X'' - \lambda^2 X = 0 \quad \text{and} \;\; T' - \lambda^2 \kappa T = 0 \\ & \text{On solving these two equations, we get} \\ & X = b_1 e^{\lambda x} + b_2 e^{-\lambda x} \quad \text{and} \quad T = b_3 e^{\lambda^2 \kappa t} \\ & \therefore \quad u(x,t) = \left(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right) e^{\lambda^2 \kappa t} \\ & \text{where} \quad c_1 = b_1 b_3, \quad c_2 = b_2 b_3 \end{aligned} \tag{21}$$

Case III. Let $\mu = -\lambda^2 \quad \lambda \neq 0$,Then solution of equations (18) and (19) are given by

$$\begin{aligned} & X'' + \lambda^2 X = 0 \quad \text{and} \;\; T' + \lambda^2 \kappa T = 0 \\ & \text{On solving these two equations, we get} \\ & X = C_1 \cos \lambda x + C_2 \sin \lambda x \quad \text{and} \quad T = C_3 e^{-\lambda^2 \kappa t} \\ & \therefore \quad u(x,t) = \left(D_1 \cos \lambda x + D_2 \sin \lambda x\right) e^{-\lambda^2 \kappa t} \\ & \text{where} \quad D_1 = C_1 C_3, \quad D_2 = C_2 C_3 \end{aligned} \tag{22}$$

We know that $u \to 0$ as $t \to \infty$. So only suitable solution of equation (15) is given by equation (22).

1.3 Solution of two dimensional Heat equation by separation of variables method

Consider two dimensional heat equation

$$u_{xx} + u_{yy} = \frac{1}{\kappa} u_t \tag{23}$$

Suppose the solution of equation (23) is of the form

$$u(x,t) = X(x)Y(y)T(t) \neq 0 \tag{24}$$

putting the value from equation (24) in equation (23)

$$\begin{aligned} & X''YT + XY''T = XYT' \\ \implies & \frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{T} \end{aligned} \tag{25}$$

Now, since $x \;\; Y$ and T are independent variables, thus equation (25) is true if each term on each side is equal to a constant such that

$$\begin{aligned} & \tfrac{X''}{X} = -n^2 \quad \tfrac{Y''}{Y} = -m^2 \quad \text{and} \;\; \tfrac{T'}{T} = -p^2 \\ & \text{with} \quad n^2 + m^2 = p^2 \\ & \lim_{t \to \infty} u \to 0 \end{aligned} \tag{26}$$

Solving equation (25), we get

$$X_n(x) = A_n\cos(nx) + B_n\sin(nx);\quad Y_m(y) = C_m\cos(mx) + D_m\cos(mx)$$
and $T_p(t) = E_p e^{-p^2\kappa t} = F_{nm}e^{-(n^2+m^2)\kappa t}$
$$\implies u_{nm}(x,y,t) = F_{nm}\Big(A_n\cos(nx) + B_n\sin(nx)\Big)\Big(C_m\cos(mx) + D_m\cos(mx)\Big)e^{-(n^2+m^2)\kappa t}$$
Hence $\quad u(x,y,t) = \Sigma^n\Sigma^m u_{nm}(x,y,t)$

is required solution of given two dimensional heat equation.

1.4 Solution of two dimensional Laplace equation by separation of variables

Consider a two dimensional Laplace equation in Cartesian coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{27}$$

Let

$$u(x,y) = X(x)Y(y) \tag{28}$$

be a solution of equation (28).

Using equations (28) in (27), we get

$$X''Y + XY'' = 0$$
$$\implies \tfrac{X''}{X} = -\tfrac{Y''}{Y} = \mu \ \ \textbf{(say)}$$

where μ is a separation constant.

Now, we have the following cases:

Case (i): Let $\mu = \lambda^2$, $\lambda \neq 0$ is real. Then

$$\tfrac{d^2X}{dx} - \lambda^2 X = 0 \qquad \text{and} \quad \tfrac{d^2Y}{dy} - \lambda^2 Y = 0$$

The solution of above equations are given by

$$X = C_1e^{\lambda x} + C_2e^{-\lambda x} \quad \text{and} \quad Y = C_3\cos\lambda y + C_4\sin\lambda y$$

Thus, in this case , the required solution is

$$u(x,y) = X(x)Y(y) = \Big(C_1e^{\lambda x} + C_2e^{-\lambda x}\Big)\Big(C_3\cos\lambda y + C_4\sin\lambda y\Big)$$

Case (ii): Let $\mu = 0$ then equations (**??**) has solution be

$$X = d_1x + d_2 \quad \text{and} \quad Y = d_3y + d_4$$

Thus, in this case , the required solution is

$$u(x,y) = (d_1x + d_2)(d_3y + d_4) \tag{29}$$

Case (iii): Let $\mu = -\lambda^2$ then same as case (i).

............**All the best**.........