Solution of Boundary value problem or Formation of Fourier Series

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1 Wave Equation

1.1 One Dimensional Wave Equation

Consider the one dimensional wave equation

$$u_{tt} = \tfrac{1}{c^2} u_{xx}, \quad c > 0, \quad t \geq 0, \quad x \in (0, l) \tag{1}$$

The initial conditions are

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

The boundary conditions are

$$u(0, t) = 0, \qquad u(l, t) = 0, \quad \forall \quad t$$

Let $u(x, t)$ be the required solution of equation (2) is of the form

$$u(x, t) = X(x)T(t) \tag{2}$$

$$\textbf{s.t.}, \quad X(x) \neq 0, \quad T(t) \neq 0, \quad \therefore \quad u(x, t) \neq 0$$

putting the value from equation (2) in equation (2)

$$X''T = \frac{1}{c^2} T''X$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{T''}{c^2 T} = \mu \quad \textbf{(say)} \tag{3}$$

where μ is separation constant.
From equation (3), we can deduce that

$$X'' - \mu X = 0 \tag{4}$$

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and

$$T'' - c^2 \mu T = 0 \tag{5}$$

Using boundary conditions and equation (2)

$$\begin{aligned} X(0)T(t) = 0 \quad &\textbf{and} \quad X(a)T(t) = 0 \\ \Rightarrow \quad X(0) = 0, \quad X(a) = 0 \quad &\quad T(t) \neq 0 \end{aligned} \tag{6}$$

Case (i.) Let $\mu = 0$, in this case, solution of equation (4) is given by

$$X(x) = Ax + B \tag{7}$$

then using (6) in (7), we get

$$\begin{aligned} &B = 0 \quad \text{and} \quad Al + B = 0 \\ &\quad \Rightarrow \quad A = 0 \quad \Rightarrow \quad X(x) = 0 \\ &u(x,t) = 0 \end{aligned}$$

which is **trivial solution**. Aim to find non-trivial solution. So solution is rejected.

Case (ii.) Let $\mu = \lambda^2$, $(\lambda \neq)$ in this case, solution of equation (4) is given by

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \tag{8}$$

then using (6) in (8), we get

$$\begin{aligned} &A + B = 0 \quad \text{and} \quad Ae^{\lambda l} + Be^{-\lambda l} = 0 \\ &\text{on solving above two equations, we get} \\ &A = B = 0 \quad \Rightarrow \quad X(x) = 0 \\ &\Rightarrow \quad u(x,t) = 0 \end{aligned} \tag{9}$$

which is also trivial solution. Hence this case also rejected

Case (iii.) Let $\mu = -\lambda^2$, $(\lambda \neq)$ Let in this case, solution of equation (4) is given by

$$X(x) = A\cos\lambda x + B\sin\lambda x \tag{10}$$

then using (6) in (10), we get

$$\begin{aligned} &A = 0 \quad \text{and} \quad A\cos\lambda l + B\sin\lambda l = 0 \\ &\text{on solving above two equations, we get} \end{aligned}$$

Now

$$\begin{aligned} &\sin\lambda l = 0 \quad [B \neq 0] \\ &\Rightarrow \quad \lambda = \tfrac{n\pi}{l}, \quad n \in \mathbb{N} \end{aligned} \tag{11}$$

Thus, the non-trivial(non-zero) solution of equation (**??**)

$$X_n(x) = B_n \sin\left(\tfrac{n\pi x}{l}\right) \tag{12}$$

Using Equations (11) and (5), we get

$$T'' + \Big(\frac{n^2\pi^2c^2}{l^2}\Big)T = 0$$

whose general solution is given by

$$\begin{aligned} &T_n(t) = c_n\cos\left(\tfrac{n\pi ct}{l}\right) + D_n\sin\left(\tfrac{n\pi ct}{l}\right)\\ &\textbf{So},\quad u_n(x,t) = X_n(x)T_n(t) = \Big[E_n\cos\left(\tfrac{n\pi ct}{l}\right) + F_n\sin\left(\tfrac{n\pi ct}{l}\right)\Big]\sin\left(\tfrac{n\pi x}{l}\right)\end{aligned} \tag{13}$$

where $E_n = c_nB_n$ and $F_n = D_nB_n$. Equation (13) is solution of equation (2) satisfying the boundary conditions.

Thus, more general solution is given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t)\\ &= \sum_{n=1}^{\infty}\Big[E_n\cos\left(\tfrac{n\pi ct}{l}\right) + F_n\sin\left(\tfrac{n\pi ct}{l}\right)\Big]\sin\left(\tfrac{n\pi x}{l}\right)\end{aligned} \tag{14}$$

The initial conditions give

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} E_n\sin\frac{n\pi x}{l}$$

which is a half-range Fourier sin series, where

$$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \tag{15}$$

From equation (14)

$$u_t(x,t) = \sum_{n=1}^{\infty}\Big[-\frac{n\pi ct}{l}E_n\cos\Big(\frac{n\pi ct}{l}\Big) + \frac{n\pi ct}{l}F_n\sin\Big(\frac{n\pi ct}{l}\Big)\Big]\sin\Big(\frac{n\pi x}{l}\Big)$$

Then from initial condition ,

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} F_n\sin\Big(\frac{n\pi x}{l}\Big)\Big(\frac{n\pi ct}{l}\Big)$$

which is also a half-range Fourier sin series, where

$$F_n = \frac{2}{n\pi cl}\int_0^l g(x)\sin\frac{n\pi x}{l}dx \tag{16}$$

Hence the required physically meaningful solution is obtained from equation (14), where E_n and F_n are given by equations (15) and (16). $u_n(x,t)$ given by equation (13) are called normal modes of vibration and $\frac{n\pi c}{l} = \omega_n$, $n = 1, 2, 3, ...$ are called normal frequencies.

1.2 Two Dimensional Wave Equation

To find solution of

$$u_{tt} = c^2\Big(u_{xx} + u_{yy}\Big),\quad c > 0,\quad t \geq 0,\quad x \in (a,b)\times(a,b) \tag{17}$$

Subject to
the initial conditions are

$$u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = g(x, y)$$

and the boundary conditions are

$$u(0, y, t) = u(a, y, t) = 0, \qquad u(x, 0, t) = u(x, b, t) = 0, \quad \forall \quad t$$

Let $u(x, t)$ be the required solution of equation (17) is of the form

$$u(x, t) = X(x)YyT(t) \tag{18}$$

$$\textbf{s.t.}, \quad X(x) \neq 0, \quad y(y) \neq 0 \text{ and } T(t) \neq 0, \qquad \therefore \quad u(x, t) \neq 0$$

then **boundary conditions** becomes

Along x-axis

$$X(0)Y(y)T(t) = 0 \quad \text{and} \quad X(a)Y(y)T(t) = 0 \quad \text{by above assumption} \tag{19}$$

we get

$$X(0) = 0 \quad \text{and} \quad X(a) = 0 \tag{20}$$

Along y-axis

$$X(x)Y(0)T(t) = 0 \quad \text{and} \quad X(x)Y(b)T(t) = 0 \tag{21}$$

we get

$$Y(0) = 0 \quad \text{and} \quad Y(b) = 0 \tag{22}$$

putting the value from equation (18) in equation (17), we get

$$X''YT + XY'' = \frac{1}{c^2}XYT''$$

$$\implies \quad \frac{X''}{X} + \frac{Y''}{Y} = \frac{T''}{c^2T} \tag{23}$$

Now, since x, y and t are independent variables, equation (23) can only be true if each term on each side is equal to a constant.
Let

$$\frac{X''}{X} = \mu_1$$

is boundary value second order O.D.E. (i.e.,)

$$\begin{aligned} &X'' - \mu_1 X = 0 \\ &X(0) = 0 \quad \text{and} \quad X(a) = 0 \end{aligned} \tag{24}$$

Case (i.) Let $\mu_1 = 0$, in this case, solution of equation (17) is given by

$$X(x) = Ax + B \tag{25}$$

then using (24), we get

$$\begin{aligned} &B = 0 \quad \text{and} \quad Al + B = 0 \\ &\Rightarrow \quad A = 0 \quad \Rightarrow \quad X(x) = 0 \\ &u(x,t) = 0 \end{aligned}$$

which is **trivial solution**.So solution is rejected.

Case (ii.) Let $\mu_1 = \lambda_1^2$, $(\lambda_1 \neq)$ in this case, solution of equation (24) is given by

$$X(x) = Ae^{\lambda_1 x} + Be^{-\lambda_1 x} \tag{26}$$

then from (24), we get

$$\begin{aligned} &A + B = 0 \quad \text{and} \quad Ae^{\lambda_1 a} + Be^{-\lambda_1 a} = 0 \\ &\text{on solving above two equations, we get} \\ &A = B = 0 \quad \Rightarrow \quad X(x) = 0 \\ &\Rightarrow \quad u(x,t) = 0 \end{aligned} \tag{27}$$

which is also trivial solution. Hence this case also rejected

Case (iii.) Let $\mu_1 = -\lambda^2$, $(\lambda \neq)$ Let in this case, solution of equation (24) is given by

$$X(x) = A\cos\lambda x + B\sin\lambda x \tag{28}$$

then using (24), we get

$$\begin{aligned} &A = 0 \quad \text{and} \quad A\cos\lambda a + B\sin\lambda a = 0 \\ &\text{on solving above two equations, we get} \end{aligned}$$

Now

$$\begin{aligned} &\sin\lambda a = 0 \quad [B \neq 0] \\ &\Rightarrow \quad \lambda = \tfrac{n\pi}{a}, \quad n \in \mathbb{N} \end{aligned} \tag{29}$$

Thus, the non-trivial(non-zero) solution of equation (24)

$$X_n(x) = B_n \sin\left(\tfrac{n\pi x}{a}\right) \tag{30}$$

Further let

$$\frac{Y''}{Y} = \mu_2$$

is boundary value second order O.D.E. (i.e.,)

$$\begin{aligned} &Y'' - \mu_2 Y = 0 \\ &Y(0) = 0 \quad \text{and} \quad Y(b) = 0 \end{aligned} \tag{31}$$

On solving boundary value problem (31) similarly as along x-axis , we get

$$Y_m(x) = D_m \sin\left(\tfrac{m\pi y}{b}\right), \quad m = 1, 2, 3, ... \tag{32}$$

where

$$\mu_2 = -\mu^2 \quad \text{and} \quad \mu = \tfrac{m\pi}{b}, \qquad m = 1, 2, 3, ...$$

Therefore, equation (23) reduces to

$$\begin{aligned} &\tfrac{T''}{c^2T} = \mu^1 + \mu_2 = -(\lambda^2 + \mu^2) = -\pi^2\left(\tfrac{n^2}{a^2} + \tfrac{m^2}{b^2}\right) \\ &\implies \quad T'' + \lambda_{mn}^2 T = 0 \end{aligned}$$

where

$$\lambda_{mn}^2 = C^2\pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \tag{33}$$

The solution of equation (36) is given by

$$T_{mn(t)} = E_{mn}\cos\lambda_{mn}t + F_{mn}\sin\lambda_{mn}t \tag{34}$$

Hence,

$$\begin{aligned} u_{mn}(x, y, t) &= X_n(x)Y_mT_{mn}(t) \\ &= \Big(A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t\Big)\sin\left(\tfrac{m\pi y}{b}\right)\sin\left(\tfrac{n\pi x}{a}\right) \end{aligned} \tag{35}$$

where $A_{mn} = E_{mn}B_nD_m$ and $B_{mn} = F_{mn}B_nD_m$ Hence, general solution is given by

$$\begin{aligned} u_{mn}(x, y, t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_n(x)Y_mT_{mn}(t) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big(A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t\Big)\sin\left(\tfrac{m\pi y}{b}\right)\sin\left(\tfrac{n\pi x}{a}\right) \end{aligned} \tag{36}$$

The initial conditions give

$$u(x, y, 0) = f(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{a}\right)$$

which is Fourier sin series, where

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x, y)\sin\frac{n\pi x}{a}\sin\frac{n\pi x}{b}dxdy \tag{37}$$

From equation (14)

$$u_t(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big(-A_{mn}\lambda_{mn}\sin\lambda_{mn}t + B_{mn}\lambda_{mn}\cos\lambda_{mn}t\Big)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi x}{a}\right)$$

Then from initial condition ,

$$u_t(x, y, 0) = g(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} K_{mn}\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi x}{a}\right)$$

where $K_{mn} = \lambda_{mn}B_{mn}$. which is also a Fourier sin series, where

$$K_{mn} = \frac{4}{ab}\int_0^a\int_0^b g(x, y)\sin\frac{n\pi x}{a}\sin\frac{n\pi x}{b}dxdy \tag{38}$$

Hence solution of equation (17) with respect to boundary as well initial condition given by equation (36), Where coefficient A_{mn} and B_{mn} given by equations (37) and(38) , respectively.

2 Heat Equation: Boundary value problem

2.1 One Dimensional Heat Equation

Consider one dimensional wave equation

$$u_t = \kappa u_{xx} \tag{39}$$

Initial condition

$$u(x, 0) = f(x) \tag{40}$$

and boundary conditions are

$$u(0, t) = 0\ , \quad u(a, t) = 0 \tag{41}$$

Solution:-

Suppose the solution of equation (39) is of the form

$$u(x, t) = X(x)T(t) \neq 0 \tag{42}$$

putting the value from equation (42) in equation (39)

$$X''T = \frac{1}{\kappa}T'X$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{T''}{\kappa T} = -\mu \quad \textbf{(say)} \tag{43}$$

where μ is separation constant.

From equation (43), we can deduce that

$$X'' + \mu X = 0 \tag{44}$$

and

$$T' + \kappa\mu T = 0 \tag{45}$$

solving equations (44) and (45), we get

Case I. when $\mu = 0$ then equations (44) with the help of 39)

$$\begin{aligned} &\Longrightarrow \quad X(x) = a_1 x + a_2 \\ &\Longrightarrow \quad a_2 = 0, \quad a_1 = 0 \\ &u(x, t) = 0 \end{aligned} \tag{46}$$

which is trivial solution. so this case is rejected.

Case II. when $\mu = -\lambda^2$ $(\lambda \neq 0)$ then equations (44)and (39)

$$\begin{aligned} &\Rightarrow \quad X(x) = b_1 e^{\lambda x} + b_2 e^{-\lambda x} \\ &\Longrightarrow b_1 + b_2 = 0 \quad \text{and} \quad b_1 e^{\lambda a} + b_2 e^{-\lambda a} = 0 \\ &\therefore \quad b_1 \ = 0, \qquad b_2 = 0 u(x, t) = 0 \end{aligned} \tag{47}$$

which is trivial solution. so this case is rejected.

Case III. when $\mu = \lambda^2$ $(\lambda \neq 0)$ then equations (44)and (39)

$$\begin{aligned} &\Rightarrow \quad X(x) = d_1 \cos \lambda x + d_2 \sin \lambda x \\ &d_1 = 0 \quad \text{ and } \ d_2 \sin \lambda a = 0 \\ &\Longrightarrow \quad \lambda = \tfrac{n\pi}{a} \qquad [d_2 \neq 0] \\ &\therefore \qquad X_n(x) = B_n \sin \tfrac{n\pi x}{a}, \qquad m = 1, 2, 3, ... \end{aligned} \tag{48}$$

Now,

$$T' + \kappa \lambda^2 T = 0$$

$$\begin{aligned} &\Longrightarrow T) = Ce^{-\lambda^2 \kappa t} \\ &\therefore \quad T_n(t) = C_n e^{-\left(\frac{n^2\pi^2}{a^2}\right)\kappa t} \end{aligned} \tag{49}$$

Therefore solution of equation (39)

$$u_n(x,t) = X_n(x)T_n(t) = B_n \sin \tfrac{n\pi x}{a} C_n e^{-\left(\frac{n^2\pi^2}{a^2}\right)\kappa t} \tag{50}$$

Hence, general solution of equation (39) is

$$\begin{aligned} &u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) \\ &= \sum_{n=1}^{\infty} E_n \sin \tfrac{n\pi x}{a} e^{-\left(\frac{n^2\pi^2}{a^2}\right)\kappa t} \end{aligned} \tag{51}$$

where $E_n = B_n C_n$ now, using initial condition E_n is given by

$$E_n = \frac{2}{a} \int_0^l f(x) \sin \frac{n\pi x}{a} dx \tag{52}$$

which is required solution of given heat equation subject to boundary as well as initial condition.

.............**All the best**.........