Nalanda Open University

M.sc Part-I

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Paper- V

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UNIT II

LATTICE THEORY

Contents : Partial Order Relation, Partially Ordered Set, Least Upper Bound, Greatest Lower Bound, Lattice, Complete Lattice, Principal Of Duality, Sub Lattice, Bounded Lattice, Homomorphism Of Lattice

1. Partial Order Relation, Partially Ordered Set, Least Upper Bound, Greatest Lower Bound, Lattice, Complete Lattice.

1.1. Introduction -: This is a study of abstract structure studies in the mathematical sub disciplines of order theory and abstract algebra.

1.2 Definitions:

Partial Order Relation:- A relation R in a set X is said to be Partial order relation if it is

(a) Reflexive if xRx holds $\forall$ x $\in$ X.
(b) Anti symmetric if xRy and yRx both holds then x = y $\forall$ x, y $\in$ X
(c) Transitive if xRy and yRz implies xRz, $\forall$ x, y, z $\in$ X
Clearly ' ≤ ' and ' $\subseteq$ ' are Partial Order relation

Partially Ordered Set:- A set X together with Partial Order relation defined on it is a Partially ordered set.

Thus the pair or the systems (X, ≤), (X, $\subseteq$) are Partially Ordered sets where ' ≤ ' and ' $\subseteq$ ' have their usual meaning .

Also two elements a and b are called comparable if either a ≤ b or b ≤ a hold otherwise they are called not comparable.

Totally Ordered Set (or Linearly Ordered set or a chain):- A Partially ordered set (X, ≤) is called a Totally ordered set if every pair of elements a, b in X are comparable in the sense that either a ≤ b or b ≤ a.

Cover of an element:- In a Partially ordered set (X, ≤) we speak that a_2 is cover of a_1 if $a_1 < a_2$ and $\exists$ no u in X such that $a_1 < u < a_2$.

Upper bound (2016):- Let A be a subset of Partially ordered set (X, ≤) then an element u in X is called an upper bound of A if a ≤ u $\forall$ a $\in$ A. However u is a least upper bound (l.u.b) if u is an upper bound of A and u ≤ v for any upper bound v of A.

Lower bound and Greatest lower bound:- A be a subset of Partially ordered set (X, ≤) then an element l in X is called a lower bound of A if l ≤ a $\forall$ a $\in$ A

Also l is called greatest lower bound (g.l.b) of A if l is a lower bound of A and l ≥ m for every lower bound m of A.

Note:- l.u.b and g.l.b are unique if they exist.

Lattice(2016) (2018):- A Partially ordered set (L, $\leq$) is called a lattice if each pair set { a, b } of elements of L has a least upper bound and a greatest lower bound in L

Also least upper bound of { a, b }is denoted by $a \vee b$. We read it as 'a cup b' or as a 'a join b' or as 'a union b' .

Similarly greatest lower bound of { a, b } is denoted by $a \wedge b$. We read it as 'a cap b' or as a 'a meet b' or as 'a intersect b'.

Also since $(a \vee b) \vee c \geq a, b, c \Rightarrow (a \vee b) \vee c$ is an upper bound for { a, b, c }

Also $(a \vee b) \vee c$ is the least upper bound (l.u.b) of { a, b, c }

Complete Lattice:- A lattice (L, $\leq$) is said to be a complete if every non empty subset (finite or infinite) of L has a l.u.b and g.l.b.

Thus if $A = \{ A_\alpha \}$ be a subset of L then

l. u. b of $A = \{ A_\alpha \}$ is denoted by $\vee A_\alpha$ and

g. l. b of $A = \{ A_\alpha \}$ is denoted by $\wedge A_\alpha$.

1.3 Theorems

Theorem (1.3) i (2016, 17,19):- Prove that Partially ordered set (P(X), $\subseteq$) is

(i) A lattice
(ii) A complete lattice

Proof :- Since P(X) $\Rightarrow$ the power set of a non empty set X = set of all subsets of X.

(1) Now let A, B $\in$ P(X) then $A \vee B = A \cup B \in P(X)$

and A, B $\in$ P(X) then $A \wedge B = A \cap B \in P(X)$

Thus (P(X), $\subseteq$) is a lattice. Thus the first part is completed.

(2) Since (P(X), $\subseteq$) is a lattice. To show (P(X), $\subseteq$) is complete.

For this, let $M = \{ A_\alpha \}$ be an arbitrary subset of P(X) then

$v A_\alpha = u A_\alpha \in P(X)$ and $\wedge A_\alpha = \cap A_\alpha \in P(X)$

Thus the lattice (P(X), $\subseteq$) is complete.

Theorem (1.3) ii):- Let X be a non empty set. Then the family of all topologies on X with respect to the relation ' is weaker than 'is (1) A lattice (2) A complete lattice.

Proof :- Let $L \subset \{ T_i \}$ be the family of all topologies on X.

(1) For any $T_i, T_j \in L$ we defined $T_i \leq T_j$ if T_i is weaker than T_j.
That is if $T_i \subseteq T_j$.
Then as we know that the relation '$\subseteq$' is Partial order relation
Hence $(L, \leq)$ is a Partially ordered relation.
Now, since $T_i \vee T_j$ = l.u.b $\{ T_i, T_j \}$ = the topology on X generated by $T_i \cup T_j$.
Similarly $T_i \wedge T_j$ = g.l.b $\{ T_i, T_j \} = T_i \cap T_j$.
Thus $T_i \vee T_j$ and $T_i \wedge T_j$ both are in L.
Therefore $(L, \leq)$ is a lattice.

(2) If $M = \{ T_{ik} \}$is any family of all topologies on X,
Then $\vee T_{ik}$ = a topology generated by $\cup T_{ik}$
Also $\wedge T_{ik}$ = a topology generated by $\cap T_{ik}$ as any intersection of topology is again a topology.
Thus the lattice $(L, \leq)$ is complete.

2. Principal of duality:-

2.1. Definition (2018) :- If S is a statement in the term of $\vee$, $\wedge$ (or any one of them) which can we defined from the set of axiom for a lattice then the dual statement S' obtained by interchanging $\vee$ and $\wedge$ can be also deduced.

Example 1:- The duel statement to $a \vee b = b \vee a$ is $a \wedge b = b \wedge a$

Example 2:- The duel statement to $a \wedge b = b \wedge a$ is $a \vee b = b \vee a$

Example 3:- The duel statement to $(a \vee b) \wedge a = a$ is $(a \wedge b) \vee a = a$ and vice versa

Example 4:- The duel statement to $a \vee a = a$ is $a \wedge a = a$

Example 5:- The duel statement to $a \wedge a = a$ is $a \vee a = a$

3. Sub Lattice, Bounded Lattice

3.1 Definition:

Sub Lattice (2018):- Let $(L, \leq)$ be a lattice. Let M be a non empty subset of L with the property that $x, y \in M \Rightarrow x \vee y$ and $x \wedge y \in M$

Then M is said to be a sub-lattice of L

Clearly every sub-lattice is a lattice with respect to the induced composition.

Bounded Lattice(2018):- A lattice L is called a bound lattice if it has greatest element 1 and a least element 0.

Properties of a bound lattice :- If L is bound lattice, then for any element $a \in L$, we have the following identities.

(1) $a \vee 1 = 1$
(2) $a \wedge 1 = a$
(3) $a \vee 0 = a$
(4) $a \wedge 0 = 0$

Example 1(2018) (of a sub-lattice):- Let F = set of all real functions defined on the closed unit interval [0, 1]

Define $f \leq g$ to mean that $f(x) \leq g(x)$ for all $f, g \in F$, $x \in [0, 1]$.

Now we taking $f \vee g$ and $f \wedge g$ such that

$(f \vee g)(x) = \max. \{ f(x), g(x) \}$

$(f \wedge g)(x) = \min. \{ f(x), g(x) \}$

Then $(F, \leq)$ is a lattice (popularly called lattice of real functions defined on [0, 1])

Let M = set of all continuous functions in F.

Then with the same Partial order relation as in F, it is easy to see that M is a sub-lattice of the lattice $(F, \leq)$

Example 2(2018) (of a sub-lattice):- Let $(L, \leq)$ be a lattice . Again let a be a fixed element of L.

Also let $M = \{ x \in L : x \leq a \}$

Since $a \leq a \ \forall a \Rightarrow a \in M \Rightarrow M$ is non empty subset of L.

Again let $x, y \in M \Rightarrow x \leq a$ and $y \leq a$.

Then $x \vee y = \text{l.u.b}\{ x, y \} \leq a$ and $x \wedge y = \text{g.l.b}\{ x, y \} \leq a$

$\Rightarrow x \vee y \in M$ and $x \wedge y \in M \Rightarrow M$ is a sub-lattice of L.

Example 3(of a Bounded lattice):- The power set P(X) of the set X under the operations of $\cup$ and $\cap$ is a bounded lattice. Since φ is the least element of P(X) and the set X is the greatest element of P(X).

Example 4(of not a Bounded lattice):- The set of positive integers I^+ under the usual ordering of $\leq$ is not a bounded lattice.

Since it has a least element 1 but the greatest element does not exist.

3.2 Theorem:-

Theorem (3.2) i :- The set of normal subgroups of a group G forms a sub-lattice of the lattice of subgroup of G.

Proof :- Let L be the set of all subgroups of a group G.

Let H, M be any two subgroups of G $\Rightarrow$ H, M $\in$ L.

We define $H \leq M$ to mean that $H \subseteq M$.

Then clearly '$\leq$' is a Partially ordered relation.

Thus (L, $\leq$) is a Partially ordered set.

Further if is defined that

$H \vee M$ = l.u.b { H, M } = subgroup generated by $H \cup M$,

$H \wedge M = H \cap M$, then $H \vee M$ and $H \wedge M$ are in L.

Thus (L, $\leq$) is a lattice.

Again, let N = set of all normal subgroups of G.

Then {e} $\in$ N, e is the identity element of G.

Thus $N \neq \emptyset$

For $N_1, N_2 \in N$ we define,

$N_1 \vee N_2$ = intersection of all normal subgroups containing N_1 and N_2

and $N_1 \wedge N_2 = N_1 \cap N_2$.

since intersection of normal subgroups is again a normal subgroup, so it follows that $N_1 \vee N_2$ and $N_1 \wedge N_2 \in N$.

Therefore N is a sub-lattice of L.

Theorem (3.2) ii :- Prove that every finite lattice L = { $a_1, a_2, ..., a_n$ } is bounded

Proof :- Since the given finite lattice is L = { $a_1, a_2, ..., a_n$ }

Since the greatest element of L is $a_1 \wedge a_2 \wedge a_3 \wedge ...... \wedge a_n$

But we know that the greatest and the least elements exist, for every finite lattice

Therefore L = { $a_1, a_2, ..., a_n$ } is bounded.

4. Homomorphism and Isomorphism of lattice:-

4.1. Introduction : Lattice Homomorphism is in fact the study of algebraic definition of a lattice in a broader way.

4.2. Definition:

Homomorphism : Let L and L′ be any two lattice then a mapping $f : L \to L'$ from L into L′ is called a Lattice Homomorphism (or simply homomorphism) if :

$f(a \vee b) = f(a) \vee f(b)$ and

$f(a \wedge b) = f(a) \wedge f(b) \;\; \forall\, a, b \in L$

Isomorphism : A Homomorphism $f : L \to L'$ is said to be isomorphism if f is one-one, onto.

Order Preserving (or Order-Homomorphism) : A mapping $f : L \to L'$ of lattice L into lattice L′ is called order preserving (or order Homomorphism) if $\forall\, a, b \in L$ we have

$a \leq b \Rightarrow f(a) \leq f(b)$. Clearly for a, b in L, f(a) and f(b) are in L′

4.3. Theorem :-

Theorem (4.3) i :- A necessary and sufficient condition for a mapping f of a lattice L onto a lattice L′ to be an isomorphism is that f and f^{-1} are both order preserving

Proof :- Firstly we suppose that the mapping $f : L \to L'$ is an isomorphism.

To show that f and f^{-1} are both order preserving.

For this, let $a, b \in L$ and $a \leq b$ then $a \vee b = b$

By assumption f is isomorphism $\Rightarrow$ f is homomorphism

Thus $f(a) \vee f(b) = f(a \vee b) = f(b)$ [since $a \vee b = b$]

That is $f(a) \vee f(b) = f(b) \Rightarrow$ f is order preserving

Again $f(a) \vee f(b) = f(a \vee b)$ [since f is homomorphism]

Thus we have

$f(a \vee b) = f(a) \vee f(b) = f(b) \Rightarrow a \vee b = b$ (as f is isomorphism so f is one-one)

That is $a \leq b$

It means $f(a) \leq f(b) \Rightarrow a \leq b$

Clearly $a = f^{-1}(f(a))$, $b = f^{-1}(f(b))$

Thus $f(a) \le f(b) \Rightarrow f^{-1}(f(a)) \le f^{-1}(f(b))$

Thus f is order preserving.

Conversely :- We suppose that $f : L \to L'$ is one-one, onto and f, f^{-1} are order preserving.

To prove that $f : L \to L'$ is an isomorphism.

For this it remains only to show that f is homomorphism.

For, let for $a, b \in L$, $c = a \vee b \Rightarrow a, b \le c \Rightarrow f(a), f(b) \le f(c)$

Now let for a moment $K \in L'$ be an arbitrary such that

$f(a), f(b) \le K$

Again due to the assumption that $K \in L'$ and f' is one-one, onto so we must get an unique d (say) in L such that $K = f(d)$

Thus clearly $f(a), f(b) \le f(d)$

But f^{-1} is order preserving so $f^{-1}(f(a)), f^{-1}(f(b)) \le f^{-1}(f(d))$

That is $a, b \le d$.

Also $c = a \vee b = \text{l.u.b}\{a, b\}$ then $c \le d \Rightarrow f(c) \le f(d) = K$

Hence f(c) is l.u.b { f(a), f(b) } so $f(a) \vee f(b) = f(c) = f(a \vee b)$

In a similar way we can see $f(a) \wedge f(b) = f(a \wedge b)$

Thus f satisfies the remaining condition to be homomorphism

Thus f is isomorphism.

Theorem (4.3) ii :- For an onto order homomorphism f on a complete lattice L, 0 and 1 are fixed points.

Proof :- Given L is complete $\Rightarrow$ L has l.u.b and g.l.b in it.

Let for a moment 1 be the l.u.b and 0 be the g.l.b of L in L.

Then for every a in L we have $a \le 1$ and $0 \le a$

Thus L contains 0 and 1 (the unit element)

Also f is order homomorphism so $0 \le a \Rightarrow f(0) \le f(a)$

Similarly $a \leq 1 \Rightarrow f(a) \leq f(1)$

Also f is onto so every element b can be expressed in the form $f(a) = b$

It means $f(0) \leq b \leq f(1) \Rightarrow f(0)$ is the zero element of L and f(1) is a unit element of L.

But we know that zero element and unit element in a lattice are unique.

Thus 0 and f(0) are not distinct.

Similarly 1 and f(1) are not distinct.

Hence $f(0) = 0$, $f(1) = 1$

That is 0 and 1 are fixed points.

Example 1:- Every finite distributive lattice is isomorphic to the lattice of lower sets of the power set of its join prime (or equivalently join irreducible) element.

Example 2:- Every distributive lattice is isomorphic to a sub-lattice of the power set lattice of some set.

5. Distributive and non Distributive lattices:-

5.1. Introduction : Study of distributive lattice relates to distributive lattice for a lattice with respect to $\wedge$ (meet) and $\vee$ (join).

5.2. Definition : A lattice L is said to be distributive if for a, b, c in L we have the following properties.

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ --------------(1)

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ ------------(2)

5.3. Theorems:-

Theorem(5.3)i:- Let L be a distributive lattice then

$\wedge$ distributives over v $\Leftrightarrow$ v distributives over $\wedge$ or equivalently both statements (1) and (2) are equivalent.

Proof :- Let us assume that $\wedge$ distributives over v

That is $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds good then we find that,

$(a \vee b) \wedge (a \vee b) = ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) = a \vee ((a \vee b) \wedge c)$

$= a \vee ((a \wedge c) \vee (b \wedge c))$

$= (a \vee (a \wedge c)) \vee (b \wedge c)$

$= a \vee (b \wedge c)$

$\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ holds.

Similarly dually $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Thus the above conditions are equivalent.

Note i:- It is clear from above that principal of duality holds also for distributive lattice.

Note ii:- To establish distributive law in any lattice L it is sufficient to show that a, b, c in L

Either $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$

Or $a \vee (b \wedge c) \geq (a \vee b) \wedge (a \vee c)$

Since we it is easy to see

Since $b \vee c \geq b \Rightarrow a \wedge (b \vee c) \geq a \wedge b$

Also $b \vee c \geq b \Rightarrow a \wedge (b \vee c) \geq a \wedge c$

Thus $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Similarly $a \vee (b \wedge c)) \leq (a \vee b) \wedge (a \vee c)$

Corollary (5.3)ii:- The lattice of ideals is an arbitrary ring is not always distributive.

Observation :- Let R be the ring of integers.

Also let L = the set of all ideals in R.

Then it is easy to see that $(L, \subseteq)$ is a partially ordered set.

We now define $I_1 \wedge I_2 = I_1 \cap I_2$ and $I_1 \vee I_2$ = ideal generated by $I_1 \vee I_2$ for $I_1, I_2 \in L$.

Thus $I_1 \wedge I_2, I_1 \vee I_2$ are in $L \Rightarrow L$ is a lattice.

That is L is a lattice of ideals.

We now verify the distributivity in L.

For this, let us take :

I_1 = ideal formed by all integral multiples of 2

I_2 = ideal formed by all integral multiples of 3

I_3 = ideal formed by all integral multiples of 5

Then $I_2 I_3 = \{ 0, \pm 15, \pm 30, \pm 45, \pm, ... \}$

Thus $I_1 \wedge (I_2 \vee I_3) = I_1 \cap (I_2 I_3) = \{ 0, \pm 30, \pm 60, \pm 90, ... \}$--------------(1)

On the other hand:

$(I_1 \wedge I_2) \vee (I_1 \wedge I_2) = \{ 0, \pm 6, \pm 12, ... \} \vee \{ 0, \pm 10, \pm 20, ... \}$

$= \{ 0, \pm 60, \pm 120, ... \}$--------------(2)

Thus from (1) and (2)

$I_1 \wedge (I_2 \vee I_3) \neq (I_1 \wedge I_2) \vee (I_1 \wedge I_3)$.

Corollary (5.3)iii:- A lattice $(Z, \leq)$ of positive integers in which $a \leq b$ is to mean that a/b (i.e, a divides b) then the lattice Z is distributive.

Observation :- Since $a \vee b$ = L. C. M. of a and b

Also $a \wedge b$ = H. C. F. of a and b

Then by number theoratic fact we know that

If H. C. F. of m and n is denoted by (m, n) and

L. C. M. of m and n is denoted by [m, n] then

$[a, (b, c)] = ([a, b], [a, c])$

That is $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Thus $(Z, \leq)$ is a distributive lattice..

6. Modular lattice (Dedekind Lattice), Interval in a lattice.

6.1. Introduction:
On a verification it is easy to see that a good number lattices studied in algebra are not distributive. However they satisfy a weaker form of distributivity which gives rise the concept of a lattice popularly called a modular lattice.

6.2. Definition:

Modular lattice: A lattice $(L, \leq)$ is said to be a modular lattice whenever $a \leq b$ then $a \vee (b \wedge c) = b \wedge (a \vee c)\ \forall\ a, b, c \in L$.

Interval in a lattice: Let L be a lattice and $a, b \in L$ then a set E containing $x \in L$ such that $E = \{ x \in L : a \leq x \leq b, x \in L\}$ is called a closed interval in L. E is denoted by I [a, b]. E is also called quotient in L.

Similar intervals: Any two closed interval I [m, n] and I [p, q] in a modular lattice are said to be similar if for any $a, b \in L$ if any one interval is expressed as $I [a, a \vee b]$ then other must be expressed as $I [a \wedge b, b]$.

Projective intervals: Any two intervals I [m, n] and I [p, q] in a modular lattice L if we can get a finite sequence of intervals such as

I [m, n] = I [m_1, n_1], I [m_2, n_2], ..., I [m_i, n_i] = I [p, q] in L.

Where consecutive pairs of intervals are similar.

6.3. Theorems:

Theorem(6.3)i:- Any sub-lattice of a modular lattice is modular.

Proof :- Let M be a modular lattice and L be its sub-lattice.

Then a, b $\in$ L we have $a \vee b$, $a \wedge b$ are in L

Let a, b, c $\in$ L and $a \leq b$ but then a, b, c $\in$ M such that $a \leq b$ and M is modular.

Thus we shall have $a \vee (b \wedge c) = b \wedge (a \vee c)$

Thus L is also modular.

Theorem(6.3)ii:- If L is a distributive lattice and for a, c $\in$ L, $a \leq c$ then L is modular.

Proof :- Since a, c are in L and $a \leq c$ then for b $\in$ L,

$a \vee (b \wedge c) = (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge c \Rightarrow$ L is modular.

Theorem(6.3)iii:- Let G be a group and L be the set of all normal(invariant) subgroups of G then L forms a modular lattice.

Proof :- Let $N_1, N_2 \in L$. Also let $N_1 \leq N_2$ to mean that $N_1 \subseteq N_2$.

Then obviously (L, $\leq$) is a partially ordered set.

Clearly $N_1 \cap N_2 \in L$ as intersection of any two normal sub-group is normal

Also $N_1 \cap N_2$ = g.l.b { N_1, N_2 } = $N_1 \wedge N_2$.

Also $N_1 \cup N_2$ is not necessarily a normal subgroup is normal.

Then $N_1 \cup N_2$ is a normal subgroup and thus $N_1 \cup N_2 \in L$

But $N_1 \cup N_2$ is the l.u.b { N_1, N_2 }

Thus for { N_1, N_2 } both the l.u.b and g.l.b of { N_1, N_2 } $\in$ L

Thus (L, $\leq$) is a lattice.

Let $N_1, N_2, N_3 \in L$ and $N_1 \leq N_2$.

$N_1 \vee N_3$ is normal subgroup generated by $N_1 \cup N_3$.

Also we have $N_1 \vee N_3 = N_1 N_3 = N_3 N_1$.

Now let $x \in N_2 \wedge (N_1 \vee N_3)$ then

$x = n_2 \in N_2$ and $x = n_1 n_3$ for $n_1 \in N_1$, $n_3 \in N_3$

Again $n_2 = n_1 n_3 \Rightarrow n_1^{-1} n_2 = n_3 \Rightarrow n_1^{-1} n_2 \in N_2$ (as $N_1 \subseteq N_2$)

$\Rightarrow x = n_1 n_3 \in N_1 \vee (N_2 \wedge N_3)$

Thus it is established that $N_2 \wedge (N_1 \vee N_2) \leq N_1 \vee (N_2 \wedge N_3)$

But by general lattice theoratic property we know that

$N_2 \wedge (N_1 \vee N_3) = N_1 \vee (N_2 \wedge N_3)$

Thus the lattice (L, $\leq$) is modular.

Theorem(6.3)iv:- Let (i) R be a ring, (ii) L = the set of ideals of R. Such that for I_1, I_2 in L

$I_1 \leq I_2 \Rightarrow I_1 \subseteq I_2$ then prove that

(a) '$\leq$' is a Partial order relation and hence (L, $\leq$) is a partially ordered set
(b) (L, $\leq$) is a lattice
(c) Lattice (L, $\leq$) is modular.

Proof :- Since given for $I_1, I_2 \in L$, $I_1 \leq I_2$ means $I_1 \subseteq I_2$.

It is a well known for that $\subseteq$ (the set inclusion relation) is a Partial order relation on L and hence (L, $\leq$) is a partially ordered set.

Thus part (a) is done

We now come to establish (b) :

For this, since $I_1 \cap I_2 \in L$ as intersection of two ideals is an ideal.

Also $I_1 \cap I_2 = \text{g.l.b} \{ I_1, I_2 \} = I_1 \wedge I_2 \Rightarrow I_1 \wedge I_2 \in L$

Also ideal generated by $I_1 \cup I_2$ denoted by $[I_1, I_2]$ is the l.u.b $\{ I_1, I_2\} = I_1 \vee I_2$ in L.

Thus (L, $\leq$) is a lattice.

We now come to establish the remaining part (c):

Let $I_1, I_2, I_3 \in L$, $I_1 \leq I_2$ then $I_1 \vee I_3 = I_1 I_3 = I_3 I_1$

So if $x \in I_2 \wedge (I_1 \vee I_3)$ then $x \in I_2$ and $x \in I_1 \vee I_3$.

Then $x = i_2 \in I_2$ and $x = i_1 i_3$ for $i_1 \in I_1$, $i_3 \in I_3$.

From $i_2 = i_1 i_3$ we get $i_1^{-1} i_2 = i_3 \Rightarrow i_1^{-1} i_2 \in I_2$ (as $I_1 \subseteq I_2$)

$\Rightarrow I_3 \in I_2 \Rightarrow i_3 \in I_2 \wedge I_3 \Rightarrow x = i_1 i_3 \in I_1 \vee (I_2 \wedge I_3)$

Thus we find that $x \in I_2 \wedge (I_1 \vee I_3) = x \in I_1 \vee (I_2 \wedge I_3)$

Therefore $I_2 \wedge (I_1 \vee I_3) \leq I_1 \vee (I_2 \wedge I_3)$.

But by general lattice theoratic property we have

$I_2 \wedge (I_1 \vee I_3) = I_1 \vee (I_2 \wedge I_3)$

Theorem(6.3) v:- Let (L, ≤) be a lattice then the necessary and sufficient for this lattice L to be modular is that, for $a, b, c \in L$,

if $a \leq b$ and $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$ implies that $a = b$

OR

State and prove the characterization theorem for modular lattice.

Proof :- First of all we assume a lattice L to be modular and $a, b, c \in L$ such that

$a \leq b$ and $a \vee c = b \vee c$, $a \wedge c = b \wedge c$.

To prove $a = b$.

For this, Since $a = a \vee (a \wedge c) = a \vee (b \wedge c) = b \wedge (a \vee c)$ (as L is modular)

$= b \wedge (b \vee c)$ (by given condition) $= b$

That $a = b$

Let $a \leq b$, $a \vee c = b \vee c$, $a \wedge c = b \wedge c \Rightarrow a = b$

To prove that lattice L is modular.

For this, we know that $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

$\leq b \wedge (a \vee c)$ (since $a \leq b$)

Also $(a \vee (b \wedge c)) \vee c = a \vee ((b \wedge c) \vee c) = a \vee c$ -------------(1)

and $a \vee c = (a \vee c) \vee c \geq (b \wedge (a \vee c)) \vee c \geq a \vee c$

But then $(b \wedge (a \vee c)) \vee c = a \vee c$ -------------------------------(2)

but by law of duality we get,

$(a \vee (b \wedge c)) \wedge c = b \wedge c$ --(3)

and $(b \wedge (a \vee c)) \wedge c = b \wedge c$ ------------------------------------(4)

on comparing (1) and (2) we get

$(a \vee (b \wedge c)) \vee c = (b \wedge (a \vee c)) \vee c \Rightarrow a \vee (b \wedge c) = b \wedge (a \vee c)$

and $(a \vee (b \wedge c)) \wedge c = (b \wedge (a \vee c)) \wedge c$ [by comparing (3) and (4)]

$\Rightarrow a \vee (b \wedge c) = b \wedge (a \vee c)$

Thus for a, b, c in L the required equality for L to be modular is obtained.

Therefore L is modular.

Corollary (6.3) vi:- A modular lattice satisfies a weaker distributive law.

Observation :- Let $(L, \leq)$ be a modular lattice.

Then for every a, b in L, $a \leq b \Rightarrow a \vee (b \wedge c) = b \wedge (a \vee c), c \in L$ ----------(1)

Also, as $a \leq b \Rightarrow a = b \wedge a$ and conversely --------------------------------------(2)

Now since L is modular so from (1) and (2) we have,

$(b \wedge a) \vee (b \wedge c) = b \wedge (a \vee c)$ is the distributive law.

Thus a modular lattice L satisfies a weaker distributive law.

Theorem (6.3) vii:- Let L be a modular lattice then the intervals $I[a, a \vee b]$ and $I[a \wedge b, b]$ are isomorphic for a and b are in L.

Proof :- Let $x \in I[a, a \vee b] \Rightarrow a \leq x \leq a \vee b \Rightarrow a \wedge b \leq x \wedge b \leq b \wedge b \leq b$

But $b \in I[a \wedge b, b]$

$\Rightarrow x \wedge b \in I[a \wedge b, b]$

Similarly if $y \in I[a \wedge b, b] \Rightarrow a \wedge b \leq y \leq b$

$\Rightarrow [(a \wedge b) \vee a] \leq y \vee a \leq b \vee a$

$\Rightarrow a \leq y \vee a \leq a \vee b$

$\Rightarrow y \vee a \in I [a, a \vee b]$

We now consider an into mapping $f : I [a, a \vee b] \rightarrow I [a \wedge b, b]$ given by

$f(x) = x \wedge b$ ------------------------------ (1)

we also consider an into mapping $g : I [a \wedge b, b] \rightarrow I [a, a \vee b]$ given by

$g(y) = y \vee a$ -------------------------------(2)

clearly f and g are well defined.

Now, let $x \in I [a, a \vee b]$

$\Rightarrow a \leq x \leq a \vee b$

$\Rightarrow x \wedge (a \vee b) = a \vee (x \wedge b)$ [since L is modular]

and $x = x \wedge (a \vee b) = a \vee (x \wedge b)$ -----------------------(3)

Thus by principal of duality

$y \in I [a \wedge b, b] \Rightarrow y = b \wedge (y \vee a)$

thus we can write the composition of f and g in the following way:

$x \rightarrow^{f} y = x \wedge b \rightarrow^{g} y \vee a = (x \wedge b) \vee a = a \vee (x \wedge b) = x$ [by (3)]

Therefore f and g are inverses of each other.

Thus either f or g is one-one onto. [but the inverse of a one-one onto mapping is also one-one onto]

Also, let $x, u \in I [a, a \vee b]$ and $a \leq x \leq u \leq a \vee b$

Also by (1) $f(x) = x \wedge b \leq u \wedge b = f(u) \Rightarrow$ f is order preserving

In a similar way we can see g is also order preserving

Thus we find that f and g are one-one, onto and order preserving

Thus f and g are isomorphism.

Theorem (6.3) viii:- State and prove cancellation law in a modular lattice L.

Statement :- Cancellation law states that : a, b, c are in a modular lattice L

Where $(a \vee b) \wedge c = 0$ then $a \wedge (b \vee c) = a \wedge b$

Proof :- Since $a \wedge (b \vee c) = a \wedge (a \vee b) \wedge (b \vee c)$ (by properties of modular lattice)

$= a \wedge [b \vee (a \vee b) \wedge c]$

$= a \wedge b$ [since by question $(a \vee b) \wedge c = 0$]

7. Complemented Lattice, Atom, Join Irreducible, Meet Irreducible

7.1. Introduction

Here we study those elements of a complemented modular lattice whose complements are unique an element with unique complement decompose the lattice into a direct product of sub-lattices

7.2. Definitions:

Atom : An element a of a Partially Ordered set with least element 0 is an atom if $0 \leq a$ and there is no x such that $0 \leq x \leq a$.

In other words we can define an atom to be an element which is minimal among the non zero elements or alternatively an element which covers the least element 0.

Atomic lattice : A lattice L is called atomic lattice if for every $a \neq 0$ of L, there exists an atom p of L such that $p \leq a$.

Complement of an element: Let L be a bounded lattice and $a \in L$. An element $b \in L$ is said to be a complement of a if $a \wedge b = 0$ and $a \vee b = 1$.

Unique complemented element: If an element has only one complement then it is called an Unique Complemented element of lattice L.

Complemented lattice : It is a bounded lattice, with least element 0 and greatest element 1, and in it every element a has a complement. That is an element b satisfying $a \vee b = 1$ and $a \wedge b = 0$

Note 1 : Complements need not be unique.

Relatively complemented lattice : Is a lattice such that every interval [c, d], viewed as a bounded lattice in its own right, is a complemented lattice.

Unique complemented lattice : If every element of bounded lattice L has an Unique Complement, then we call L an uniquely complemented lattice.

Join Irreducible : An element a in a lattice L is called Join irreducible if and only if a is not bottom element, and whenever $a = b \vee c$, then $a = b$ or $a = c$.

Meet Irreducible : An element a in a lattice L is said to be meet irreducible if and only if a is not a top element of L and whenever $a = b \wedge c$, then $a = b$ or $a = c$.

Irreducible element : If an element a in a lattice L is both join and meet irreducible then a is said to be simply irreducible.

Note 2 : Any atom in L is join irreducible.

Note 3 : Clearly join irreducible and meet irreducible are dual statement.

7.3. Theorems:

Theorem (7.3) i:- Every complemented modular lattice is relatively complemented.

Proof :- Let L be a complemented modular lattice.

Again let a, b be any two elements of L such that $a \leq b$

Let a' stands for the complement of a in L then by definition we know that $a \vee a' = 1$, $a \wedge a' = 0$.

Also L is modular $\Rightarrow b = b \wedge (a \vee a') = a \vee (b \wedge a') = a \vee a_1$ where $a_1 = b \wedge a'$

Again since $a \wedge a_1 = a \wedge b \wedge a' = a \wedge a' \wedge b = 0 \wedge b = 0$

Which makes clear that a_1 is complement of a relative to b.

Theorem (7.3) ii:- Let L be a bounded distributive lattice. Then complements are unique if they exist.

Proof :- Let L be a bounded distributive lattice.

Also let x, y are complements of a in L. Then $a \vee x = 1$, $a \vee y = 1$ and $a \wedge x = 0$, $a \wedge y = 0$

By distributive of L : $x = x \vee 0 = x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y) = 1 \wedge (x \vee y) = x \vee y$

Similarly, $y = y \vee 0 = y \vee (a \wedge x) = (y \vee a) \wedge (y \vee x) = 1 \wedge (y \vee x) = y \vee x$

Thus $x = x \vee y = y \vee x = y$.

That the complements are not distinct.

So the complement of an element a in L is unique.

Theorem (7.3) iii:- If L is finite complemented distributive lattice then prove that every element in L can be written uniquely as the join irreducible atom of L.

Proof :- Let $a \in L$ be arbitrary.

Also since L is finite and distributive so we can write

$a = b_1 \vee b_1 \vee .... \vee b_n$ --------------------------(1)

where b_i's are irreducible join irreducible element of L.

Also L is bounded as it is finite.

We know that in a bounded distributive, the complements are unique, if they exist.

Since L is complemented so this implies that every element in L must have unique complement.

Therefore join irreducible element of L, other than 0, is an atom of L.

Theorem (7.3) iv:- If a lattice L is complemented then every join – irreducible element is an atom.

Proof :- Let L be a complemented lattice. Then for each x in L we must have y in L such that $x \vee y = 1$, $x \wedge y = 0$

Further let $z \in L$ is a join irreducible of L.

To prove that z is an atom.

If possible let for a moment that z is not an atom.

On the other hand let x be an atom in [0, z].

Also by supposition y is the complement of x.

Then $(x \vee y) \wedge z = 1 \wedge z = z = (x \wedge z) \vee (y \wedge z) = x \vee (y \wedge z)$

Since z is join irreducible so we must have $y \wedge z = z \Rightarrow y \geq z$

But then $y > x$ and $y \wedge x = x \neq 0$ is a contradiction.

Thus supposing that join irreducible z is not an atom we arrived at a contradiction. Therefore join irreducible element z is an atom. Also z is an arbitrary element.

Hence every join irreducible elements are atom.

8. Solved Examples:

Example 1: Partially ordered set which is not a lattice

Solution : Let X = { 1, 2, 3, 4, 5, 6, 7, 8 }

Again let for $a, b \in X$, $a \leq b$ to mean that a divides b.

Then clearly $\leq$ is a partial order relation and hence $(X, \leq)$ is a partially ordered set.

But we find that $3 \vee 5$ = l.u.b $\{ 3, 5\}$ = L.C.M of 3 and 5 = $15 \notin X$

Thus the Partially ordered set $(X, \leq)$ is not a lattice.

Example 2(of an incomplete lattice): Let us consider the set N of all natural numbers.

Solution : Let us assume for m, n in N, $m \leq n$ to mean that m divides n

Then clearly $\leq$ is a partial order relation and hence $(N, \leq)$ is a partially ordered set.

Also $m \vee n$ = least common multiple of $\{ m, n\}$

and $m \wedge n$ = greatest common divisor of $\{ m, n\}$

Then clearly $m \vee n$, $m \wedge n$ are in $(N, \leq)$

Thus $(N, \leq)$ is a lattice.

But we know that N does not have an upper bound.

Thus the partially ordered set $(N, \leq)$ is a lattice but not complete.

Example 3: Let L be a lattice. Also $a, b \in L$ and $a \leq b$ then

$M = \{ x \in L : a \leq x \leq b \}$ is a sub lattice of L.

Example 4: Two similar intervals are isomorphic.

Solution : See the answer of theorem { 6.3 (vi) }.

Example 5: A complemented lattice is a bounded lattice.

Solution : Since a complemented lattice L is bounded as it contains least element 0 and greatest element 1. Also in L every element a has a complement. That is L contain an element b satisfying $a \vee b = 1$, $a \wedge b = 0$.

Also complement need not be unique.