Compatibility system and Charpit's Method

Binod Kumar*

M.Sc. Mathematics   Part: *II*

Paper-11   Mathematical Method

Nalanda Open University, Patna

1 Pfaffian Differential Equations

By a pfaffian differential equation, we mean a differential equation of the form

$$F_1(x_1,...,x_n)dx_1 + F_2(x_1,...,x_n)dx_2 + ... + F_n(x_1,...,x_n)dx_n = 0 \tag{1}$$

where F_i's are continuous function. The left of above equation is called a-pfaffian differential equation form.

A pfaffian differential equation is said to be exact, if we can find continuously differential function $u(x_1,...,x_n$ such that

$$du = F_1(x_1,...,x_n)dx_1 + F_2(x_1,...,x_n)dx_2 + ... + F_n(x_1,...,x_n)dx_n \tag{2}$$

A pfaffian differential equation is said to be integrable if $\exists$ a non-zero differential function $\mu(x_1,...,x_n)$ such that the pfaffian differential equation form

$$\mu[F_1(x_1,...,x_n)dx_1 + F_2(x_1,...,x_n)dx_2 + ... + F_n(x_1,...,x_n)dx_n] \tag{3}$$

is exact. The function $\mu(x_1,...,x_n)$ is called integrating factor and $u(x_1,...,x_n = c$,where c is an arbitrary constant, is called the integral of the corresponding pfaffian differential equation.

Theorem 1. *There always exists an integrating factor for a-pfaffian differential equation in two variables.*

Proof. Do yourself. □

Lemma Let $u(x,y)$ and $v(x,y)$ be two function of x and y such that $\frac{\partial v}{\partial y} \neq 0$.

If, further

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 \tag{4}$$

then,$\exists$ a relation

$$F(u,v) = 0 \tag{5}$$

*Corresponding author, e-mail:binodkumararyan@gmail.com, Telephone: +91-9304524851

between u and v not involving explicitly.

proof: Since $\frac{\partial v}{\partial y} \neq 0$, we can eliminate y between $u(x, y)$ and $v(x, y)$,obtain the relation

$$F(u, v, x) = 0 \tag{6}$$

On differentiate w.t.to x and y, we get, respectively

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0 \tag{7}$$

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{8}$$

On eliminating $\frac{\partial F}{\partial v}$ from these equations, which is possible $\frac{\partial v}{\partial x} \neq 0$, we find that

$$\frac{\partial F}{\partial u}\frac{\partial(u, v)}{\partial(x, y)} + \frac{\partial F}{\partial x}\frac{\partial v}{\partial y} = 0 \tag{9}$$

i.e., $\frac{\partial F}{\partial x}\frac{\partial v}{\partial y} = 0 \quad \Longrightarrow \quad \frac{\partial F}{\partial x} = 0$. (Suppose $\frac{\partial v}{\partial x} = 0$then $\quad \frac{\partial v}{\partial x} = 0(?)$),
F is independent of x.

Theorem 2. *Lemma: If $\vec{X} \cdot (\nabla \times \vec{X}) = 0$,where $\vec{X} = (P, Q, R)$ and μ is an arbitrary differentiable function of x, y and z then $\mu\vec{X} \cdot (\nabla \times (\mu\vec{X})) = 0$*

Proof. Consider

$$\mu\vec{X} \cdot (\nabla \times (\mu\vec{X})) \begin{cases} = \sum_{x,y,z}(\mu P)\left[\frac{\partial(\mu R)}{\partial y} - \frac{\partial(\mu Q)}{\partial z}\right] \\ = \ \mu^2 \sum_{x,y,z} P\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] - \mu \sum_{x,y,z}\left[PR\frac{\partial \mu}{\partial y} - PR\frac{\partial \mu}{\partial z}\right] \\ = \mu^2 \sum_{x,y,z} P\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] \\ = \mu^2(\vec{X} \cdot (\nabla \times \vec{X})) \end{cases} \tag{10}$$

Conversely, if $\vec{X} \cdot (\nabla \times \vec{X}) = 0$ then $\mu^2(\vec{X} \cdot (\nabla \times \vec{X})) = 0$ for $\mu \neq 0$ □

Theorem 3. Necessary and sufficient condition *A necessary and sufficient that the pfaffian differential equation*

$$\vec{X} \cdot \vec{dr} = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz \tag{11}$$

be integrable is that

$$\vec{X} \cdot (\nabla \times \vec{X}) = P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\Big) = 0 \tag{12}$$

Proof.

- **Necessary condition.** For, if above equation is integrable,then $\exists$ differential functions $\mu(x, y, z)$ and $u(x, y, z)$ such that

$$du = u(x, y, z)[P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz] \tag{13}$$

where $\mu(x, y, z \neq 0$. However

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \tag{14}$$

comparing these two equation, we get $\mu\vec{X} = \nabla u$. Since $\nabla \times (\nabla u) = 0$,we have $\mu\vec{X} \cdot (\nabla \times \mu\vec{X}) = 0$

- **sufficient condition.** We treated z as constant $(dz = 0)$ then pfaffian equation becomes always integrable in two variables x and y, this has solution of the form $U(x, y, z) = c_1$ may be involves z,there must exists a integrating factor $\mu(x, y, z \neq 0$ such that

$$\frac{\partial U}{\partial x} = \mu P,\ \frac{\partial U}{\partial y} = \mu Q \tag{15}$$

On multiplying both side of pfaffian equation μ and then substituting the above,we get

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz + (\mu R - \frac{\partial U}{\partial z})dz \tag{16}$$

This implies that

$$dU + Kdz = 0,\ where\ k = (\mu R - \frac{\partial U}{\partial z}) \tag{17}$$

hence by Lemma $\mu\vec{X} \cdot (\nabla \times (\mu\vec{X})) = 0$ observe that

$$\mu\vec{X} = (\mu P, \mu Q, \mu R) = \begin{cases} \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right) \\ \nabla U + \left(0, 0, K\right) \end{cases} \tag{18}$$

Hence

$$\mu\vec{X} \cdot \nabla(\mu\vec{X}) = \begin{cases} \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right) \cdot \left(\frac{\partial K}{\partial y}, -\frac{\partial U}{\partial x}, 0\right) \\ \frac{\partial(U,K)}{\partial(x,y)} \end{cases} \tag{19}$$

Therefore

$$\frac{dU}{dz} + K(U, z) = 0 \tag{20}$$

it posses a solution $\phi(U, z) = c_1$. it can be expressed in the form $u(x, y, z) = c$

□

Example 1. *Show that the pfaffian equation is integrable and find integral* $ydx + xdy + 2zdz = 0$

Solution: *Since* $\nabla \times \vec{X} = 0$, *where* $\vec{X} = (y, x, 2z)$. *Hence pfaffian equation is exact* $ydx + xdy + 2zdz = d(xy + z^2) = 0$.*Therefore* $u(x, y, z) = xy + z^2 = c$

Example 2. *Find integral* $yzdx + 2zxdy - 3xydz = 0$

solution: Since $\vec{X} \cdot (\nabla \times \vec{X}) = 0$,*where* $\vec{X} = (yz, 2xz, -3xy)$ *and* $U = xy^2 = c_1$. *Then* $\mu = \frac{y}{z}$. *Further,* $K = \frac{y(-3xy)}{z} = -\frac{3U}{z}$. *Therefore* $\frac{dU}{dz} - \frac{3U}{z} = 0$,*hence integral* $u(x, y, z) = \frac{xy^2}{z^3} = c$

2 Compatible System of First order partial differential equation

Definition: The equation

$$f(x, y, z, p, q = 0) \tag{21}$$

and

$$g(x, y, z, p, q = 0) \tag{22}$$

are compatible on a domain D if

$$(i) \quad j = \frac{\partial(f,g)}{\partial(p,q)} \neq 0, \; on \; \mathbf{D} \tag{23}$$

$$(ii) \quad p = \phi(x,y,z), \; q = \psi(x,y,z) \tag{24}$$

where

$$dz = \phi(x,y,z)dx + \psi(x,y,z)dy, \quad is \; integrable \tag{25}$$

Theorem 4 (The Necessary and Sufficient for Integrability)**.** *A necessary and sufficient condition for the integrability of* $dz = \phi(x,y,z)dx + \psi(x,y,z)dy$ *is*

$$[f,g] \equiv \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} \tag{26}$$

Proof. Given equation is integrable if and only if

$$\vec{X} \cdot (\nabla \times \vec{X}) = 0, \; where \; \vec{X} = (\phi, \psi, -1) \tag{27}$$

i.e.,$-\phi(-\psi_z + \psi(\phi) - (\psi_x - \phi_y) = 0$

$$\psi_x + \phi\psi_z = \phi_y) + \psi\phi_z \tag{28}$$

On substituting ϕ and ψ for p and q respectively in Eq. (21) and differentiating it w.r.to x and z,we obtain

$$f_x + f_p\phi_x + f_q\psi_x = 0 \tag{29}$$

$$f_z + f_p\phi_z + f_q\psi_z = 0 \tag{30}$$

On multiplying the second Eq. by ϕ and adding it to first,we get

$$f_x + f_z\phi + f_p(\phi_x + \phi\phi_x) + f_q(\psi_x + \phi\psi_z) = 0 \tag{31}$$

similarly, for Eq.(22) that

$$g_x + g_z\phi + g_p(\phi_x + \phi\phi_x) + g_q(\psi_x + \phi\psi_z) = 0 \tag{32}$$

Solving these equations, we find that

$$\psi_x + \phi\psi_z = \frac{1}{J}\Big(\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}\Big) \tag{33}$$

If we differentiate the given pair of equations w.r.to y and z, we obtain

$$\phi_x + \psi\phi_z = -\frac{1}{J}\Big(\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\Big) \tag{34}$$

Using above result E, we get required result is that $[f,g] = 0$. A solution is of the form $F(x,y,z,c)$,where c is an arbitrary constant. Therefore system is compatible then they have a one-parameter family of common solutions. □

Note: Compatibility can also be shown by verifying the condition $[f, g] = 0 \quad and \ \frac{\partial(f,g)}{\partial(p,q)} \neq 0$.

Example 1. *Show that the equations $f = xp - yq - x = 0$ and $g = x^2p + q - xz = 0$ are compatibility and also find solutions.*

Solution: Since $\frac{\partial(f,g)}{\partial(p,q)} = x(1 + xy) \neq 0$ on the domain D where $D \subseteq \{(x, y) \in \mathbb{R}^2, \ \ni x \neq 0, \ xy + 1 \neq 0\}$. Then on D ,we obtain

$$p = \frac{1 + yz}{1 + xy}, \ q = \frac{x(z - x)}{1 + xy}, \ then \ dz = \frac{1 + yz}{1 + xy}dx + \frac{x(z - x)}{1 + xy}dy \tag{35}$$

$$then \ \frac{dz - dx}{z - x} = \frac{ydx + xdy}{1 + xy}, \quad \Rightarrow \quad z = x + c(1 + xy) \tag{36}$$

Remarks: For a compatible system not necessary every solution of $f = 0$ is solution of $g = 0$ and vice-versa. Above example $z = x(1 + y)$ is solution of $f = 0$ but not $g = 0$.

Charpit's method In this, we present a method to find complete integral of first order p.d.e. Let

$$f(x, y, z, p, q) = 0 \tag{37}$$

By compatibility $\exists$ a family of p.d.equations.

$$g(x, y, z, p, q, a) = 0, \ such \ that \ [f, g] = 0 \tag{38}$$

$$dg = f_p\frac{\partial g}{\partial x} + f_q\frac{\partial g}{\partial y} + (pf_p + qf_q)\frac{\partial g}{\partial z} - (f_x + pf_z\frac{\partial g}{\partial p}) - (f_y + qf_z)\frac{\partial g}{\partial q} = 0 \tag{39}$$

This is quasi-linear p.d.e. for g with x, y, z, p and q, then Charpit's auxiliary equations be

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + pf_z} = \frac{dg}{0} \tag{40}$$

Find out the value of either $p(\text{or}q)$ from Charpit's equation and then put this value in given p.d.e to get value of $q(\text{or}p)$. Then values of p and q substitute in $dz = pdx + qdy$ then integrate.

Example 2. *Find a complete integral of $f = z^2 - pqxy = 0$ by Charpit's method.*

Solution:

The Charpit's auxiliary equations of the given p.d.e

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = -\frac{dp}{2zp - pqy} = -\frac{dq}{2zq - pqx} \tag{41}$$

then

$$\frac{dz}{2z^2} = \frac{pdx + qdy + xdp + ydq}{2z(px + qy)}$$

$$\Rightarrow z = a(xp + qy) \tag{42}$$

which is compatible with $f = 0$ is $g(x, y, z, p, q, a) = z - a(xp + qy) = 0$. Further solving for p and q from $f = 0$ and $g = 0$, we obtain $p = \frac{z}{cx}$,putting the value in $f = 0$, we get $q = \frac{cz}{cy}$,where $a(c + \frac{1}{c}) = 1$, then

$$\begin{aligned} dz &= \frac{z}{cx}dx + \frac{cz}{cy}dy \\ \Rightarrow z &= bx^{\frac{1}{c}}y^{c} \end{aligned} \tag{43}$$

...............All the best.........