M.Sc. Mathematics, Part—I

Paper—I (Advanced Abstract Algebra)

Unit—I

Composition Series

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Normal subgroup—A subgroup H of a group G is called a normal Subgroups of G if for every x Є G and for every h in H, $x\ h\ x^{-1}$ Є H or equivalently x h = h x for every x in G and every h in H.

Note—1:

A normal subgroup is also called an invariant subgroup.

Note—2:

Every Abelian group is normal. Every subgroup of an Abelian group is normal.

Note—3

Every group G has at least two normal subgroups namely the group G itself and the subgroup consisting of the identity element e alone. These two normal subgroups are known as improper normal subgroups of G.

Cyclic group => Abelian group=> Normal => Every Cyclic group or sub group is Normal.

Simple Group— A group $G \neq \{e\}$ is called a simple group if it has no proper normal subgroup of G.

Maximal Normal subgroup—A normal subgroup N of a group G is said to be maximal iff ∃ no Normal Subgropu K of G such that $N \subset K \subset G$. I.e. N is not contained in K properly and K is not contained in G properly.

Composition series for a group G-- Let G be a group and H_1, H_2, H_3........ H_n are the subgroups of G then the finite sequence $G = H_1, H_2, H_3 \ldots\ldots H_n = \{e\}$ is called a composition series for G provided each H_i except H_1 is a maximal normal subgroup of H_{i-1}.

Subnormal series— Let H_0, H_1, H_2,……..H_n be a finite sequence of subgroups of a group G then the sequence is called a subnormal series if H_i is the subset of H_{i+1} and H_i is a normal subgroup of H_{i+1} with $H_0 = \{e\}$ and $H_n = G$.

It is also known as sub invariant series.

Normal Series-- A normal series of a group G is a finite sequence H_0, H_1, H_2…..H_n of normal subgroups such that H_i is the $\subseteq H_{i+1}$, $H_0 = \{e\}$, $H_n = G$.

Note—1:

As every subgroup of an Abelian group is Abelian so this is why the concept of Subnormal and normal series coincides for Abelian group.

Note—2:

Every normal series is subnormal but the converse is not necessary

Refinement—a Subnormal (or normal) series $\{K_J\}$ is a refinement of subnormal (or normal) series $\{H_i\}$ of a group G if $\{H_i\}$ is a subset of $\{K_J\}$.

I.e. if each H_i is one of the K_j

Equivalently the subnormal series G contains $K_1 \supset K_2 \supset............\supset K_n = \{e\}$ is called a refinement of the subnormal series

Note—1:

Every subnormal series is a refinement of itself.

Composition Series—a subnormal series that has no refinement other than itself is known as a composition series.

Composition series of group G—A subnormal series $G = G_0 \supset G_1 \supset G2....... \supset G_m = \{e\}$ is said to be composition series of G if all the factors of the series G_i/G_{i-1} are simple groups.

Example (1):- Let G = {z} be the group of integers under addition, then $\{0\} < 8z < 4z < z$ and $\{0\} < 9z < z$ are subnormal series of z.

Example (2):- If we take into consideration Dihedral group D_4 of symmetries of the square then we get a series $\{P_0\} < \{P_0, \mu_1\} < \{P_0, P_2, \mu_1, \mu_1\} < D_4$ is a subnormal series of D_4.

Also the series is not a normal series because $\{P_0, \mu\}$ is not normal in D4.

Example (3):- A series $\{0\} < 72z < 24z < 8z < 4z < z$ is a refinement of the series $\{0\} < 72Z < 8Z < Z$

Factors of the Subnormal (or Normal) Series—Let G be a group then the factor groups (or quotient groups) G/H_1, H_1/H_2..........$H_{m-1}/\{e\}$ are called the factors of the subnormal series: $G = H_0 \supset H_1 \supset H_2 \supset..........Hm = \{e\}$ of G.

Length of the Series—the number of factors in a subnormal (or normal) series $H_0 \supset H_1 \supset H_2 \supset.......H_m = \{e\}$ is called the length of the series. Clearly here the length of the above series is m.

Isomorphism of any two subnormal (or normal) Series—Let H_i & K_j be any two subnormal (or normal) series of a group G then {Hi} and $\{K_J\}$ are called isomorphic, if there is a one to one correspondence between the collection of factor groups $\{H_{i+1}/ H_i\}$ and $\{K_{J+1}/K_J\}$ such that corresponding factor groups are isomorphic.

Clearly two isomorphic subnormal (normal) series must have the same number of groups.

Example—If $G = Z_{15}$ group of integers addition module 15, then, the two series of G $G \supset \{5\} \supset \{0\}$ are isomorphic because $Z_{15}/5$ and $(3)/\{0\}$ are isomorphic to Z_5 and $Z_{15}/(3)$ is isomorphic to $(5)/\{0\}$.

State and prove Zassenhaus Lemma—

This lemma states that if:

(A) H and K are subgroups of a group G
(B) $H^*$and K^* are normal subgroups of H and K respectively, then:

(a) $H^*(H \cap K^*)$ is a normal subgroup of $H^*(H \cap K)$
(b) $K^*(H^* \cap K)$ is a normal subgroup of $K^*(H \cap K)$
(c) $H^*(H \cap K)/ H^*(H \cap K^*) \equiv K^*(H \cap K)/ K^*(H^* \cap K) \equiv (H \cap K)/ [(H^* \cap K) (H \cap K^*)]$

Proof—

Let for a moment suppose that $C= H \cap K$, $D = (H \cap K^*) (K \cap H^*)$ then obviously $D \subseteq C$.

By assumption K* is normal in K and $C= H \cap K$ => C is a subgroup of K.

Thus $C \cap K^* = H \cap K \cap K^* = H \cap K^*$ is normal in $H \cap K$.

Also the product of two normal subgroups is again normal.

Hence $H^*(H \cap K^*)$ is a normal in $H^*(H \cap K)$……….. (1)
Interchanging H and K we get $K^*(K \Pi H^*)$ is normal in $K^*(K \cap H^*)$............................ (2)
Also $D = (H \cap K^*) (K \cap H^*)$ is normal in C.

So we get a factor group D in C and for a moment let it be L.

Then as we know that $L = C/D = H \cap K / (H \cap K^*) (K \cap H^*)$.

Also $H^*C = H^*(H \Pi K)$ is a subgroup as H* is normal in H.

Also if a*c = H*C => a*Є H and c Є C Then L = {DC: c Є C} (3)

Similarly if a1* Є H and c1 Є C then a*c = a1* c1 = $a1^{*-1}$ a* = c1 $c1^{-1}$.

But $H^* \cap C = H^* \cap H \cap K$ is the subset of $H^* \cap K$ then c1 $c1^{-1}$ Є $H^* \cap C$ is the subset of $H^* \cap K \subseteq D$.

=> $a1^{*-1}$ a* is the subset of D => ($a1^{*-1}$ a*). C Є Dc

=> C1 Є Dc => c1 Є L by equation (3)

Thus we get a mapping f: H*C→ F of H*C into F.

Since every element c of C is mapped onto its cosets DC So f is onto F. Also f is homomorphic.
Also H* is normal in H*C => (a1* c1) (a2* c2) = a3* (c1 c2) for some a3* Є H* remains to show that Ker (f) = $H^* (H \cap K^*)$

For, since $H \cap K^*$ CD

If f maps G*C then C is in D.

But D = (H ∩ K*) (K ∩ H*), then we have

Let u Є (H ∩ K*) and v Є (K ∩ H*) and c = u.v then

a*c = a*(uv) = (a*u)v = a1*v for some a1* X H* => a*c = a1*v Є H* (H ∩ K*) Thus Ker(f) = the subgroup H*(H ∩ K*)

Thus by a theorem on homomorphism, we get—

H* (H ∩ K)/ H*(H ∩ K*) is equivalent to (H ∩ K)/ (H ∩ K*)(K ∩ H*)………(4)

Similarly we can get K*(K ∩ H) / K*(K ∩ H*) is equivalent to K ∩ H/ (K Π H*)(H ∩ K*)…..(5)

Thus from (4) and (5) we can say that

H*(H ∩ K)/ H*(H ∩ K*) is equivalent to K*(K ∩ H) / K*(K ∩ H*)

SCHRIER THEOREM—

State and Prove Schrierer Theorem:

This theorem states that any two subnormal series of an arbitrary group have isomorphic refinement or any two subnormal series of an arbitrary group have equivalent refinements.

Proof: - Let a group G has the following two subnormal series:-

$\{e\} = H_0 \subset H_1 \subset H_2 \subset ...... \subset H_n = G$………. (1) And

$\{e\} = K_0 \subset K_1 \subset K_2 \subset ...... \subset K_m = G$……….(2)

Let (1) and (2) form the chain of groups for i ($0 \le i \le n-1$) such that

$H_i = H_i (H_{i+1} \cap K_0)$ is the subset of $H_i (H_{i+1} \cap K_1)$ is the subset of $H_i (H_{i+1} \cap K_m) = H_{i+1}$

We now insert m-1 groups, not necessarily distinct, between H_i and H_{i+1},

If it is done for each i where $0 \le i \le n-1$.

Also let $H_{iJ} = H_i(H_{i+1} \Pi K_J)$ then we get the chain of groups

$\{e\}=H_{0,o} \subseteq H_{0,1} \subseteq H_{0,2} \subseteq ....\subseteq H_{0,m-1} \subseteq H_{1,o}$

$\subseteq H_{1,1} \subseteq H_{1,2} \subseteq ……... \subseteq H_{1,m-1} \subseteq$ of $H_{2,o}$

$\subseteq H_{2,1} \subseteq H_{2,2} \subseteq ……. \subseteq H_{2,m-1} \subseteq H_{3,o}$

$\subseteq$…………………………………………….

$\subseteq H_{n-1,1} \subseteq H_{n-1,2} \subseteq …….\subseteq H_{n-1,m-1} \subseteq H_{n,m}$

= G.. (3)

The above chain of groups contains nm+1 groups (not necessarily distinct) and $H_{i, 0} = H_i$ for each i.

By Zassenhaus lemma (3) is subnormal series. It means each group is normal in the following group. This chain refines the series (1)

By symmetry we get,

$K_{J,i} = K_j (K_{j+1} \cap H_i)$, for $0 \le j \le m-1$, $0 \le i \le n$.

This gives the following subnormal chain

$\{e\} = K_{0,o} \subseteq K_{0,1} \subseteq K_{0,2} \subseteq \ldots \subseteq K_{0,n-1} \subseteq K_{1,o}$

$\subseteq K_{1,1} \subseteq K_{1,2} \subseteq \ldots\ldots\ldots\ldots\ldots\ldots \subseteq K_{1,n-1} \subseteq K_{2,o}$

$\subseteq K_{2,1} \subseteq K_{2,2} \subseteq \ldots\ldots\ldots\ldots\ldots\ldots\ldots \subseteq K_{2,n-1} \subseteq K_{3,o}$

$\subseteq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\subseteq K_{m-1,1} \subseteq K_{m-1,2} \subseteq \ldots\ldots\ldots\ldots \subseteq K_{m-1,n-1} \subseteq K_{m-1,n} = G \ldots\ldots\ldots (4)$

This chain contains mn+1 groups which are not necessarily distinct and $K_{j,o} = K_j$, for each j.

This chain refines series (2).

Again by Zessenhaus lemma, we have,

$H_i (H_{i+1} \cap K_{j+1})/ H_i (H_{i+1} \cap K_j)$ is equivalent to $K_j (K_{j+1} \cap H_{i+1})/ K_j (K_{j+1} \cap H_i)$ or, $H_{ij+1} / H_{ij} = K_{ji+1} / K_{ji}$ for $0 \leq i \leq n-1$, $0 \leq j \leq m-1$…………………(5)

The isomorphism of (5) gives a one to one correspondence of isomorphic factor groups between the subnormal chain given by (3) and (4).

For verification, let us note that:

$H_{i,0} = H_i$ and $H_{i,m} = H_{i+1}$ while $K_{j,o} = K_j$ and $K_{j,n} = K_{j+1}$

Corresponding to the factor group arising from the r^{th} column of subsets in chain (4).

We after deleting repeated groups from the chain in (3) and (4), we get subnormal series of distinct groups that are isomorphic refinements of chain (1) and (2).

Examples on composition series:-

(1) Let S = {1, 2, 3}, G = S_3 = the symmetric group of degree 3
Then S3 = {I, (2 3), (1 2), (1 2 3), (1 3 2), (1 3)}

Then G contains {I, (1 2 3), (1 3 2)} contains {I} is a composition series of G = S3

(2) Let G be a cyclic group generated by a of order 6.
That is G = $\{a, a^2, a^3, a^4, a^5, a^6 = e\}$ then,

G, $\{e, a^3\}$, {e} and G, $\{e, a^2, a^4\}$, {e} are two different composition series for G.

Theorem (1)—

There exists at least one composition series for every finite group G.

Proof: - Let G be a finite group

We have to consider here two cases—

Case—i: If G is simple then G, {e} is a composition series.

Case—ii: If G is not simple then there will exists a proper normal subgroup say H of G. If H is maximal in G and {e} is maximal in H then, G, H, {e} is a composition series.

If H is not maximal in G, but {e} is maximal in H then there will be a normal subgroup K of G such that H C K C G.

Now if K is maximal in G and H is maximal in K then G, H, {e} is a composition series.

Again if H is maximal in G but {e} is not maximal in H then there will exist a normal subgroup J of H such that {e} C J C H.

Then if {e} is maximal in J and J is maximal in H then G, H, J, {e} is a composition series.

Again if H is not maximal in G, {e} is not maximal in H then there exist a normal subgroup L of G such that H C L C G.

Also there exist a normal subgroup N of H such that {e} C N C H.

Thus {e} C N C H C L C G.

If L is maximal in G, H is maximal in L, N is maximal in H and {e} is is maximal in N, then G, L, H, N, {e} is a composition series.

Also since G is finite so there will be only a finite number of subgroups of G. Thus we shall get a composition series.

Theorem—2 Any two composition series of a group G are isomorphic.

Proof—

Let G be a group and it has the following two composition series:-

{e} = Gm C Gm-1 C .... C G2 C G1 C G0 = G -------------------(1) and

{e} = Hn C Hn-1 C……C H2 C H1 C H0 = G……………….. (2)

Since (1) and (2) are subnormal series of G.

Therefore by Shreirer's theorem series (1) and (2) have isomorphic refinements. So they are equivalent => m = n. In this situation neither of the series has any further refinements. So they are isomorphic.

Theorem—3

If a group G has a composition series and H is a proper normal subgroup of G then there exists a composition series containing H.

Proof—

By hypothesis G is a group and H is a proper normal subgroup of G. Thus H ≠ {e} or H ≠ G => {e} C H C G……….. (1) Is both a subnormal and normal series.

By question G has a composition series.

Let this series be g = $G_0 \supset G_1 \supset G_i \supset ...... \supset G_m = \{e\}$……. (2)

But then by Shreirer's theorem the refinement of (1) is isomorphic to the refinement of (2). But (2) is a composition series so it will not have any further refinement. Thus (1) is the required composition series.

Theorem—4

If a group G is commutative and has a composition series then G is finite.

Proof—

Let {e} = Gm C Gm-1 C.... C G2 C G1 C G0 = G be a composition series of G. Since G is abelian so its each subgroup Gi is necessarily normal.

Thus as we know that the quotient (or factor) group

G_{m-1} / G_m is equivalent to $G_{m-1} /\{e\}$ is equivalent to G_{m-1} is simple and abelian.

But we know by a theorem that a simple abelian group is cyclic group of prime order.

Thus O (Gm-1) must be prime number.

Let O (Gm-1) = Pm-1, a prime number. Also Gm-2 / Gm-1 must be prime number.

Also G_{m-2} / G_{m-1} must be prime number.

Let O (G_{m-2} / G_{m-1}) = P_{m-2}.

Hence the number of co-sets of G_{m-1} in G_{m-2} is P_{m-2}.

Also G_{m-1} has P_{m-1} element.

Thus O (G_{m-2}) = $P_{m-1} . P_{m-2}$

Thus continuing the same process, we can get

O (G) = P0. P1……….. Pm-1. Pm-2 = a product of finite number = finite number

Thus G is a finite group.

Solved Problems:

Problem 1: Find isomorphic refinement of the subnormal series {o} C 8 Z C 4 Z C Z....... (1) and {o} C 9 Z C Z.............. (2)

Solution: We consider the refinement

{o} C 72 Z C 8 Z C 4 Z C Z of (1) and the refinement {o} C 72 Z C 81 Z C 9 Z C Z of (2).

Clearly refinements of series (1) and (2) have the same length.

So they have four factors group as follows:

Z/4 Z ≡Z_4≡18 Z/72 Z

4 Z/8 Z ≡Z_2 ≡ 9 Z/18 Z

8 Z/72 Z ≡ Z_9 ≡ Z/9 Z

And 72 Z/ {O} ≡ 72 Z

Thus the refinements of (1) and (2) are the required isomorphic refinements of the given subnormal series.

Problem 2: Show that the group of integers Z has no composition series.
Solution: Let Z = G0 ⊃ G1 ⊃ G2 ⊃....... ⊃ Gn-1 ⊃ Gn= {O}

Be a subnormal series of Z and G_{n-1} must be of the form rZ for some positive integer r.

Now since Gn-1/Gn is equivalent to rz and rZ is infinite cyclic group having many non trivial proper normal subgroups of Z.

Thus Gn-1 / Gn is not simple.

Therefore Z has no composition series.

Problem 3: Find all composition series of Z60

Solution: We know that Z_{60} represents the group of integers under addition module 60.

Clearly, the sets (2),(3),(4),(5),(6),(10),(12),(15),(20),(30) are all subsets of Z_{60}.

Thus all composition series of Z60 are as follows:-

{O} C (12) C (4) C (2) C Z_{60} ;{ O} C (12) C (6) C (2) C Z_{60}
{O} C (12) C (6) C (3) C Z_{60} ;{ O} C (20) C (4) C (2) C Z_{60}
{O} C (20) C (10) C (2) C Z_{60} ;{ O} C (30) C (6) C (2) C Z_{60}
{O} C (30) C (10) C (2) C Z_{60} ;{ O} C (30) C (6) C (3) C Z_{60}
{O} C (30) C (15) C (3) C Z_{60} ;{ O} C (20) C (10) C (5) C Z_{60}
{O} C (30) C (10) C (5) C Z_{60} ;{ O} C (30) C (15) C (5) C Z_{60}

Problem 4:

Show that the series {e} C A_n C S_n of S_n is a composition series S_n, where S_n is a symmetric group and A_n is an alternating group.

Solution: It is known that the factor group A_n /{e} is isomorphic to A_n, which is simple for all n ≥ 5 and the factor group S_n / A_n is isomorphic to Z_2 , which is simple. Hence the series {e} C A_n C S_n ia a composition series of S_n.

Problem 5: Find all composition series of $Z_5 * Z_5$.

Solution—Obviously the composition series are of the form $Z_5 * Z_5$ contains H contains {(0,0)} where H may be any of the subgroups {(0,1)} , {(1,0)} , {(1,1)} , {(1,2)} , {(1,3)} and {(1,4)} of $Z_5 * Z_5$. Thus there are six in all

Z_5 x $Z_5 \supset$ {(0, 1)} ⊃ {(0, 0)}, Z_5 x $Z_5 \supset$ {(1, 0)} ⊃{(0, 0)}

Z_5 x $Z_5 \supset$ {(1,1)} ⊃ {(0,0)} ,Z_5 x $Z_5 \supset${(1,2)} ⊃ {(0,0)}

Z_5 x $Z_5 \supset${(1, 3)} ⊃{(0, 0)} and $Z_5 * Z_5 \supset${(1, 4)} ⊃ {(0, 0)}