SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Lecture 2: (a) Gauss Elimination in matrix notation.
(b) Simple partial pivoting.

<u>Gauss Elimination Method (in matrix notation)</u>

In matrix-form, the system of linear equations can be written in the form

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

where $A = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{2n}^{(1)} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}^{(1)} & a_{n2}^{(1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{nn}^{(1)} \end{pmatrix}$ is an (n $\times$ n) matrix, $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$ is a (n $\times 1$) matrix of unknowns and $\underset{\sim}{b} = \begin{pmatrix} b_1^{(1)} \\ b_2^{(1)} \\ \ldots \\ b_n^{(1)} \end{pmatrix}$ is a (n $\times 1$) matrix of prescribed constants. The augmented matrix is given by

$$\left(A, \underset{\sim}{b}\right) = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} \ldots\ldots\ldots\ldots a_{1n}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \ldots\ldots\ldots\ldots a_{2n}^{(1)} & b_2^{(1)} \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ a_{n1}^{(1)} & a_{n2}^{(1)} \ldots\ldots\ldots\ldots a_{nn}^{(1)} & b_n^{(1)} \end{pmatrix}$$

If $a_{11}^{(1)} \neq 0$, then multiplying the first row by $m_{i1} = \frac{-a_{i1}^{(1)}}{a_{11}} \left(i = 2, 3, \ldots\ldots\ldots, n\right)$, called the **multiplier,** we add them to the corresponding elements of the i$^{\text{th}}$ row $\left(i = 2, 3, \ldots\ldots\ldots, n\right)$, which means that the new values are gives as follows:

$$\text{new}\, a_{ij} = \text{old}\, a_{ij} - m_{i1} a_{1j}$$

$$\text{new}\, b_i = \text{old}\, b_i - m_{i1} b_1 \,,$$

$$a_{i1} = 0, \;\; i = 2, 3, \ldots\ldots\ldots\ldots, n \,;\; j = 2, 3, \ldots\ldots\ldots\ldots, n\,.$$

Thus, the new augmented matrix is given by

$$\begin{pmatrix} a_{11}^{(1)} \ a_{12}^{(1)} \ldots\ldots\ldots\ldots\ldots a_{1n}^{(1)} & b_1^{(1)} \\ 0 \ a_{22}^{(2)} \ldots\ldots\ldots\ldots\ldots a_{2n}^{(2)} & b_2^{(2)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ 0 \ a_{n2}^{(2)} \ldots\ldots\ldots\ldots\ldots a_{nn}^{(2)} & b_n^{(2)} \end{pmatrix}$$

If $a_{22}^{(2)} \neq 0$, then we multiply the second row by the multiplier $m_{i2} = -\dfrac{a_{i2}^{(2)}}{a_{22}^{(2)}} \ (i = 3, 4, \ldots\ldots\ldots, n)$ and add them to the corresponding element of the i[th] row $(i = 3, 4, \ldots\ldots\ldots\ldots\ldots, n)$ and like before,

$$\text{new}\, a_{ij} = \text{old}\, a_{ij} - m_{i1} a_{1j}$$

$$\text{new}\, b_i = \text{old}\, b_i - m_{i1} b_1 \ ,$$

$$a_{i2} = 0, \ i = 3, \ldots\ldots\ldots\ldots\ldots, n; \ j = 3, \ldots\ldots\ldots\ldots\ldots, n .$$

The process is repeated and after $(n-1)$ steps we get the augmented matrix as

$$(A, \underset{\sim}{b}) = \begin{pmatrix} a_{11}^{(1)} \ a_{12}^{(1)} a_{13}^{(1)} \ldots\ldots\ldots\ldots\ldots a_{1n}^{(1)} & b_1^{(1)} \\ 0 \ a_{22}^{(2)} a_{23}^{(2)} \ldots\ldots\ldots\ldots\ldots a_{2n}^{(2)} & b_2^{(2)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ 0 \quad 0 \quad 0 \ldots\ldots\ldots\ldots\ldots 0 \ a_{nn}^{(n)} & b_n^{(n)} \end{pmatrix}$$

and we can get the value of the unknowns $x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_n$ by the method of back substitution.

Example 1. Solve the system of linear equations using Gauss Elimination method:

$$0.4x + 5y + 2z = 7.4$$
$$6x - 0.3y + 4z = 9.7$$
$$-5x + 2y + 0.8z = -2.2$$

Solution: In matrix form, the equations can be written as $A\underset{\sim}{x} = \underset{\sim}{b}$, where

$$A = \begin{pmatrix} 0.4 & 5 & 2 \\ 6 & -0.3 & 4 \\ -5 & 2 & 0.8 \end{pmatrix}, \tilde{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} 7.4 \\ 9.7 \\ -2.2 \end{pmatrix}$$

The augmented matrix is given by

$$\left(A, \underset{\sim}{b}\right) = \begin{pmatrix} 0.4 & 5 & 2 & | & 7.4 \\ 6 & -0.3 & 4 & | & 9.7 \\ -5 & 2 & 0.8 & | & -2.2 \end{pmatrix} \qquad (a_{11} = 0.4 \neq 0. \text{ We divide the first row by } 0.4)$$

$$\approx \begin{pmatrix} 1 & 12.5 & 5 & | & 18.5 \\ 6 & -0.3 & 4 & | & 9.7 \\ -5 & 2 & 0.8 & | & -2.2 \end{pmatrix}, \qquad \left(\text{Notation}: R_1' = \frac{1}{0.4} R_1\right)$$

$$\approx \begin{pmatrix} 1 & 12.5 & 5 & | & 18.5 \\ 0 & -75.3 & -26 & | & -101.3 \\ 0 & 64.5 & 25.8 & | & 90.3 \end{pmatrix} \qquad \left(R_2' = R_2 - 6R_1,\ R_3' = R_3 + 5R_1\right)$$

$$\approx \begin{pmatrix} 1 & 12.5 & 5 & | & 18.5 \\ 0 & 1 & 0.3453 & | & 1.3453 \\ 0 & 64.5 & 25.8 & | & 90.3 \end{pmatrix} \qquad \left(R_2' = \frac{-1}{75.3} R_2\right)$$

$$\approx \begin{pmatrix} 1 & 12.5 & 5 & | & 18.5 \\ 0 & 1 & 0.3453 & | & 1.3453 \\ 0 & 0 & 3.52815 & | & 3.52815 \end{pmatrix} \qquad \left(R_3' = R_3 - 64.5R_2\right)$$

The pivotal equations are

$$x + 12.5y + 5z = 18.5$$
$$y + 0.3453z = 1.3453$$
$$3.52815z = 3.52815$$

By back substitution we get, z = 1, y = 1, x = 1.

Example 2. Solve the system of linear equations by Gauss Elimination method:

$$\begin{aligned} x+2y+3z &= 10 \\ x+3y-2z &= 7 \\ 2x-y+z &= 5 \end{aligned}$$

<u>**Solution:**</u> In matrix form, the equations can be written as $A\underset{\sim}{x}=\underset{\sim}{b}$, where

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & -2 \\ 2 & -1 & 1 \end{pmatrix},\ \tilde{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b=\begin{pmatrix} 10 \\ 7 \\ 5 \end{pmatrix}.$$

The augmented matrix is given by

$$\left(A, \underset{\sim}{b}\right)=\begin{pmatrix} 1 & 2 & 3 & | & 10 \\ 1 & 3 & -2 & | & 7 \\ 2 & -1 & 1 & | & 5 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 2 & 3 & | & 10 \\ 0 & 1 & -5 & | & -3 \\ 0 & -5 & -5 & | & 15 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 2 & 3 & | & 10 \\ 0 & 1 & -5 & | & -3 \\ 0 & 0 & -30 & | & -30 \end{pmatrix}$$

The pivotal equations are

$$\begin{aligned} x+2y+3z &= 10 \\ y-5z &= -3 \\ -30z &= -30 \end{aligned}$$

By back substitution, we get z = 1, y = 2 and x = 3.

Simple Partial Pivoting

Suppose $a_{11}^{(1)} = 0$. In that case, $a_{11}^{(1)}$ cannot be used as a pivotal element, that is, it is not possible for this number to eliminate other elements of the first column. Again, if at any step, the pivotal element is very small in magnitude, then the corresponding multiplier is very large numerically. Multiplying the pivoted equation by such large value increase sound off errors and other computational errors in the coefficients and the constants.

Simple partial pivoting takes care of all these problems. In this process, from all the equations, we choose the one with numerically largest coefficient of x_1 and name it as the first pivotal equation, with non-zero pivot $a_{11}^{(1)} \neq 0$. The order of other equations are kept arbitrary. We then eleminate x_1 from the last $(n-1)$ linear equations using the pivotal equation and obtain a system of $(n-1)$ linear equations in $(n-1)$ unknowns, namely, $x_2, x_3, ..., x_n$. We then look for the numerically largest coefficient of x_2 (non-zero) in the reduced system of $(n-1)$ linear equations and name it as the second pivotal equation. The same process is repeated at every subsequent step.

Example 3. Solve the system of linear equation by Gauss Elimination method:

$$0.006x + 3y + 2z = 12.006$$
$$x + 0.5y + 2z = 8$$
$$2x + 4y + 0.7z = 12.1$$

Solution: In matrix-form, the equations can be written as $A\underset{\sim}{x} = \underset{\sim}{b}$, where

$$A = \begin{pmatrix} 0.006 & 3 & 2 \\ 1 & 0.5 & 2 \\ 2 & 4 & 0.4 \end{pmatrix}, \tilde{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} 12.006 \\ 8 \\ 12.1 \end{pmatrix}$$

The augmented matrix is

$$\left(A, \underset{\sim}{b}\right) = \begin{pmatrix} 0.006 & 3 & 2 & | & 12.006 \\ 1 & 0.5 & 2 & | & 8 \\ 2 & 4 & 0.7 & | & 12.1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 500 & 333.33 & | & 2000.5 \\ 1 & 0.5 & 2 & | & 8 \\ 2 & 4 & 0.7 & | & 12.1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 500 & 333.33 & | & 2000.5 \\ 0 & 499.5 & -331.33 & | & 1992.5 \\ 0 & -996 & -665.96 & | & -3988.9 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 500 & 333.33 & | & 2000.5 \\ 0 & 1 & -0.6633 & | & 3.989 \\ 0 & -996 & -665.96 & | & -3988.9 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 500 & 333.33 & | & 2000.5 \\ 0 & 1 & -0.6633 & | & -3.989 \\ 0 & 0 & -1326.61 & | & -7961.944 \end{pmatrix}$$

The pivotal equations are

$$x + 500y + 333.33z = 2000.5$$
$$y - 0.6633z = -3.989$$
$$-1326.61z = -7961.944$$

By back-substitution we get z = 6.002, y = – 0.008, z = 3.8, whereas the true solution is x = 1, y = 2, z = 3. This has happened because the pivotal element 0.006 is too small relative to the other coefficients.

We now use (simple) partial pivoting to obtain the solution, that is, we interchange first and third equation's, avoiding 0.006 for pivotal element. The third equation is chosen because the coefficient of x is 2, greater than the other two equations. Thus, the new augmented matrix is

$$\left(A, \underset{\sim}{b}\right) = \begin{pmatrix} 2 & 4 & 0.7 & | & 12.1 \\ 0.006 & 3 & 2 & | & 12.006 \\ 1 & 0.5 & 2 & | & 8 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 2 & 0.35 & | & 6.05 \\ 0.006 & 3 & 2 & | & 2.006 \\ 1 & 0.5 & 2 & | & 8 \end{pmatrix} \qquad \left(R_1^1 = \frac{1}{2} R_1\right)$$

$$\approx \begin{pmatrix} 1 & 2 & 0.35 & | & 6.05 \\ 0 & 2.988 & 1.9979 & | & 11.9697 \\ 0 & -1.5 & 1.65 & | & 1.95 \end{pmatrix} \qquad \begin{pmatrix} R_2^1 = R_2 - 0.006 R_1 \\ R_3^1 = R_3 - R_1 \end{pmatrix}$$

We next look for the largest numerical values for the y-coefficient of the reduced system, which 2.988 in this case. Hence, there is no need of interchanging the second and the third row.

$$\left(A, \underset{\sim}{b}\right) \approx \begin{pmatrix} 1 & 2 & 0.35 & | & 6.05 \\ 0 & 2.988 & 1.9979 & | & 11.9697 \\ 0 & 0 & 2.653 & | & 7.959 \end{pmatrix}$$

The pivotal equations are

$$x + 2y + 0.35z = 6.05$$
$$2.988y + 1.9979z = 11.9697$$
$$2.653z = 7.959$$

By back-substitution, we get, z = 3, y = 2 and x = 1 (true solution), which shows the usefulness of partial pivoting.

Exercises

1. Solve, by Gauss Elimination method, the system

$$x + 3y + 2z = 5$$
$$2x - y + z = -1$$
$$x + 2y + 3z = 2$$

(Ans: $x = 1, y = 2, z = -1$)

2. Solve the following system of equations by Gauss-Elimination method, correct to three places of decimals:

(a)

$$5.091x + 3.455y + 1.091z = 1.276$$
$$2.818x + 6.455y - 4.273z = 4.654$$
$$1.273x - 3.091y + 7.545z = 2.187$$

(Ans: $x = -1.992, y = 2.751, z = 1.753$)

(b)

$$1.660x + 0.684y + 0.820z + 0.380\omega = -4.925$$
$$0.784x + 1.690y + 1.396z + 0.492\omega = 6.105$$
$$0.754x + 1.602y + 1.608z + 0.456\omega = 7.325$$
$$0.442x + 0.570y + 0.338z + 1.398\omega = -4.175$$

(Ans: $x = -6.069, y = 2.929, z = 5.502, \omega = -3.592$)

(3) Solve , by Gauss-Elimination method, the system

$$0.003x + 4.00y + 5.00z = 9.003$$
$$-3.00x + 3.85y - 6.75z = -5.900$$
$$4.00x - 5.25y - 3.50z = -4.750$$

Explain why the solution deviates from true solution $(1,1,1)^T$. Use simple partial pivoting and solve the system again. Did you see any difference in the solutions?