Nalanda Open University

M.SC Part-1

Course : Physics

Paper : 1

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Topic- Mathematical Physics (Eigen value and Eigen vector of Matrix)

Eigen Values and Eigen Vectors

Eigen vector of a matrix A is a vector represented by a matrix X such that when X is multiplied with matrix A, then the direction of the resultant matrix remains same as vector X.

Mathematically, above statement can be represented as:

$$AX = \lambda X$$

where A is any arbitrary matrix, λ are eigen values and X is an eigen vector corresponding to each eigen value.

Here, we can see that AX is parallel to X. So, X is an eigen vector.

We know that,

$$AX = \lambda X$$

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow (A - \lambda I)\, X = 0 \;.....(1)$$

Above condition will be true only if $(A - \lambda I)$ is singular. That means,

$|A - \lambda I| = 0$ …..(2)

(2) is known as characteristic equation of the matrix.

The roots of the characteristic equation are the eigen values of the matrix A.

Now, to find the eigen vectors, we simply put each eigen value into (1) and solve it by Gaussian elimination, that is, convert the augmented matrix $(A - \lambda I) = 0$ to row echelon form and solve the linear system of equations thus obtained.

Consider the example

$$\begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}.$$

The characteristic equation is

$$\det\begin{pmatrix} -2-\lambda & -4 & 2 \\ -2 & 1-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{pmatrix} = 0.$$

Expanding the determinant,

$$(-2-\lambda)\,[(1-\lambda)\,(5-\lambda)-2\times 2]+4[(-2)\times(5-\lambda)-4\times 2]+2[(-2)\times 2-4\,(1-\lambda)]=0.$$

Expanding the brackets and simplifying:

$$-\lambda^3+4\lambda^2+27\lambda-90=0,$$

or, equivalently

$$\lambda^3-4\lambda^2-27\lambda+90=0.$$

By trial and error, we find that

$$3^3-4\times 3^2-27\times 3+90=0,$$

and it follows from the Factor Theorem that $(\lambda-3)$ is a factor. Indeed,

$$\lambda^3-4\lambda^2-27\lambda+90=(\lambda-3)\,(\lambda^2-\lambda-30)$$

and

$$\left(\lambda-3\right)\left(\lambda^{2}-\lambda-30\right)=\left(\lambda-3\right)\left(\lambda+5\right)\left(\lambda-6\right),$$

meaning that the eigenvalues are 3, -5 and 6.

We now go on to solve

$$\begin{pmatrix} -2-\lambda & -4 & 2 \\ -2 & 1-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for each eigenvalue λ. Now, every such system will have infinitely many solutions, because if $\mathbf{e}$ is an eigenvector, so is any multiple of $\mathbf{e}$. So our strategy will be to try to find the eigenvector with $X = 1$.

Eigenvector corresponding to eigenvalue 3

In the case $\lambda = 3$, we have

$$\begin{pmatrix} -5 & -4 & 2 \\ -2 & -2 & 2 \\ 4 & 2 & 2 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Setting $X = 1$ gives, as our first two equations,

$$-5-4Y+2Z=0,$$

$$-2-2Y+2Z=0.$$

Subtracting the first from the second:

$$3 + 2Y = 0,$$

and thus $Y = -\frac{3}{2}$.

Substituting back into the second equation,

$$-2 + 3 + 2Z = 0,$$

giving $Z = -\frac{1}{2}$.

Checking in the third equation,

$$4 - 3 - 1 = 0,$$

which works. This gives us the eigenvector

$$\left(1, -\frac{3}{2}, -\frac{1}{2}\right).$$

For convenience, we can scale up by a factor of 2, to get

$$(2, -3, -1).$$

Eigenvector corresponding to eigenvalue -5

In the case $\lambda = -5$, we have

$$\begin{pmatrix} 3 & -4 & 2 \\ -2 & 6 & 2 \\ 4 & 2 & 10 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Setting $X = 1$ gives, as our first two equations,

$$3 - 4Y + 2Z = 0,$$

$$-2 + 6Y + 2Z = 0.$$

Subtracting the first from the second:

$$-5 + 10Y = 0,$$

and thus $Y = \frac{1}{2}$.

Substituting back into the second equation,

$$-2 + 3 + 2Z = 0,$$

giving $Z = -\frac{1}{2}$.

Checking in the third equation,

$$4 + 1 - 5 = 0,$$

which works. This gives us the eigenvector

$$\left(1, -\frac{1}{2}, \frac{1}{2}\right).$$

Once again, we can scale up by a factor of 2, to get

$$(2, -1, 1).$$

Eigenvector corresponding to eigenvalue 6

In the case $\lambda = 6$, we have

$$\begin{pmatrix} -8 & -4 & 2 \\ -2 & -5 & 2 \\ 4 & 2 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Setting $X = 1$ gives, as our first two equations,

$$-8 - 4Y + 2Z = 0,$$

$$-2 - 5Y + 2Z = 0.$$

Subtracting the first from the second:

$$6 - Y = 0,$$

and thus $Y = 6$.

Substituting back into the second equation,

$$-2 - 30 + 2Z = 0,$$

giving $Z = 16$.

Checking in the third equation,

$$4 + 12 - 16 = 0,$$

which works. This gives us the eigenvector

$$(1, 6, 16).$$

General considerations

In general, the eigenvalues of a real 3 by 3 matrix can be

(i) three distinct real numbers, as here;

(ii) three real numbers with repetitions;

(iii) one real number and two conjugate non-real numbers.