ROOTS OF NON-LINEAR EQUATIONS

Lecture 3: Fixed Point Iteration.

5. Method of Iteration or Fixed –Point Iteration

Fixed point iteration method is based on the principle of finding a sequence $\{x_n\}$, each element of which successively approximates a real root α of equation $f(x) = 0, x$ in $[a, b]$. We re-write $f(x) = 0$ as $x = \emptyset(x)$. Thus, a root α of the given equation satisfies $\alpha = \emptyset(\alpha)$. Therefore, the point α remains fixed under the mapping $\emptyset$ and so a root of the equation is a fixed point of $\emptyset$.

$\emptyset\,(x)$ is called the iteration function. Here we also assume that $\emptyset(x)$ is continuously differentiable in [a, b].

Using graphical or tabulation method, we first find a location or crude approximation $[a_0, b_0]$ of a real root α say, of $f(x) = 0$ and let $x = x_0$ $[a_o \leq x_o \leq b_o]$ be the initial approximation of α. Thus α satisfies the equation

$$\alpha = \emptyset(\alpha).$$

Putting $x = x_0$ in $x = \emptyset(x)$, we get first approximation $x_1$of α as

$$x_1 = \emptyset(x_0)$$

And then the successive approximations are calculated as

$$x_2 = \emptyset(x_1),$$
$$x_3 = \emptyset(x_2),$$
$$x_4 = \emptyset(x_3),$$
$$\dots \dots \dots.$$
$$x_n = \emptyset(x_{n-1}),$$
$$x_{n+1} = \emptyset(x_n).$$

The above iteration is generated by the formula $x_{n+1} = \emptyset\,(x_n)$ and is called iteration formula, where x_n is the n^{th} approximation of the root α of $f(x) = 0$. The sequence $\{x_n\}$ of iterations or the successive better approximations may or may not converge to a limit. If $\{x_n\}$ converges, then it converges to α and the number of iterations required depend upon the desired degree of accuracy of the root α.

6. Convergence of Method of Iteration

The presentation of $f(x) = 0$ as $x = \emptyset(x)$ is not unique, therefore, the convergence of $\{x_n\}$ depends upon the nature of $\emptyset(x)$. Now, we investigate about the nature of $\emptyset(x)$ which yields a convergent sequence $\{x_n\}$.

We have, by Lagrange's mean value theorem,

$$|\alpha - x_1| = |\emptyset(\alpha) - \emptyset(x_0)| = |\alpha - x_0||\emptyset'(\varepsilon_1)| \quad \text{for } x_0 < \varepsilon_1 < \alpha$$
$$|\alpha - x_2| = |\emptyset(\alpha) - \emptyset(x_1)| = |\alpha - x_1||\emptyset'(\varepsilon_2)| \quad \text{for } x_1 < \varepsilon_2 < \alpha$$
$$|\alpha - x_3| = |\emptyset(\alpha) - \emptyset(x_2)| = |\alpha - x_2||\emptyset'(\varepsilon_3)| \quad \text{for } x_2 < \varepsilon_3 < \alpha$$

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$$|\alpha - x_{n+1}| = |\emptyset(\alpha) - \emptyset(x_n)| = |\alpha - x_n||\emptyset'(\varepsilon_n)| \quad \text{for } x_n < \varepsilon_n < \alpha.$$

Thus, $|\alpha - x_{n+1}| = |\alpha - x_o|.|\emptyset'(\varepsilon_1)|\,|\emptyset'(\varepsilon_2)|\ldots\ldots|\emptyset'(\varepsilon_n)|$

Assuming $|\emptyset'(x)| \le \rho$ in $(a_o \le x \le b_0)$, we have

$$|\alpha - x_{n+1}| \le |\alpha - x_0|\rho^n$$

Thus,

$$\lim_{n\to\infty}|\alpha - x_{n+1}| \le |\alpha - x_o| \lim_{n\to\infty}\rho^n \to 0 \text{ if } \rho < 1, \text{i.e.}, |\emptyset'(x)| < 1$$
$$\to \infty \text{ if } \rho > 1, \text{i.e.}, |\emptyset'(x)| > 1$$

Therefore,

$$\lim_{n\to\infty} x_{n+1} \to \alpha, \text{ if and only if } |\emptyset'(x)| \le \rho < 1 \text{ in } [a_0, b_0].$$

Example 6: Find the root of $x^2 + \ln(x) - 2 = 0$, which lies between 1 and 2, by fixed-point iteration method, correct to four decimal places.

Solution: Let $f(x) = x^2 + \ln x - 2 = 0$. Now, $f(1) = -1 < 0, f(2) = 2.69 > 0$. Therefore, one root lies between 1 and 2.

We write the equation as:

$$x = \sqrt{2 - \ln x} = \phi(x), \qquad \varphi'(x) = -\frac{1}{2x\sqrt{2 - \ln x}}$$

$\therefore |\varphi'(x)| < 1$ as Max $[|\varphi'(1)|, |\varphi'(2)|]$ = Max [0.35, 0.21] = 0.35 < 1.

$\therefore x = \sqrt{2 - \ln x} = \phi(x)$ gives us a convergent sequence of iteration.

We take $x_0 = 1$.

	n	x_n	$\phi(x_n)$
	0	1	1.4
	1	1.4	1.29
	2	1.29	1.32
	3	1.32	1.312
	4	1.312	1.3147
	5	1.3147	1.3139
	6	1.3139	1.31415
	7	1.31415	1.31408
	8	1.31408	1.3410
Check	9	1.31410	1.31409

Thus, **1.3141** is the root of the equation between 1 and 2, correct up to four decimal places.

Example 7. Find the root of the equation $3x - \cos x - 1 = 0$, by the iteration method, correct to four significant figures.

Solution. Let $f(x) = 3x - \cos x - 1$. As $f(0) = -2 < 0, f(1) = 1.45 > 0$, there is one root of f (x)= 0 between 0 and 1.
We re-write the equation as

$$x = \frac{\cos x + 1}{3} = \emptyset(x), \therefore \emptyset'(x) = -\frac{\sin x}{3}.$$

Therefore, $|\emptyset'(x)| < 1$, as $|\sin x| < 1$ and the convergence criteria is satisfied.

As a initial value, we take $x_0 = 0$. The successive approximations of the root are computed in tabular form as follows:

n	x_n	$\emptyset(x_n)$
0	0	0.6
1	0.6	0.61
2	0.61	0.606
3	0.606	0.6073
4	0.6073	0.60706
5	0.60706	0.60711
6	0.60711	0.60710

Thus, 0.6071 is a root of the equation, correct up to four significant figures.

Exercise 5: Find a root of the equation tan(x) + x = 0 by fixed point iteration method, correct to three significant figures. (Ans. 2.03)

Exercise 6: Find a root of the equation $10^x + x - 4 = 0$ by fixed point iteration method, correct to four significant figures. (Ans. 0.5392)