Nalanda Open University

M.Sc. Part- I

Course- Mathematics

Paper- 1 (Advanced Abstract Algebra)

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Unit- 03 MODULE

Contents:

Section 1

Module, Submodule, Union and intersection of submodules

1.1 **Introduction**: Module is in fact more closed to the concept of vector space whereas basically vector space is more closed to Abelian group. Based on this one can say that the module can be considered as the generalization of abelian group. In module we take scalar element from a ring while in vector space scaler element are from the field considered.

1.2 In this section we give some of the relevant definitions.

Module: Let R be a ring. A R- module (Left) is an additive abelian group. M together with a function R×M → M, for every r, s ∈ R and a,b ∈ M, such that the following conditions are satisfied :

(i) r (a+b) =ra+rb
(ii) (r+s)a = ra+sa
(iii) r(sa) = (rs)a

Unitary R-module:

If the ring R has an identity element then the module is called Unitary R module.

Submodule:

A nonempty subset S of R – module M is said to be a submodule of the module M if it satisfies the following conditions

(i) S is an additive subgroup of M and
(ii) ra ∈ S for r is in R and a is in S

1.3 In this section we give some of the theorems:

Theorem 1.3 (I) : Let M be a module over a ring R. Then prove that :

(i) R.0= 0, ∀ r ∈ R
(ii) 0.a= 0, ∀ a ∈ M
(iii) (-r)a = -(ra), ∀ r ∈ R, a ∈ M
(iv) (-r) (-a) = ra-rb, Ɐr ∈ R, a ∈ M
(v) R (a-b) = ra-rb , ∀ r ∈ R , a, b ∈ M
(vi) (r-s) a = ra-sa , ∀ r, s ∈ R, a ∈ M

Proof 1:

Since r.0= r(0+0) = r0 + r0

Or 0+r.0 = r.0 +r0

Hence 0=r0

Proof 2:

0.a = 0, ∀ a ∈ M

We have, 0.a =(0+0)a =0.a+0a

Or, 0+0a =0a+0a

0= 0a

Proof 3:

(-r)a+ra =[(-r) + r]a=0.a=0

So, (-r)a = -r a

Proof 4:

(-r) (-a) = ra

Since (-r)(-a) = + (-r) [=(-a)]

= -r(-a) =ra

Proof 5:

r (a-b) = r [a +(-b)]

= ra + r (-b)

= ra – rb

Proof 6:

(r-s)a = ra-sa

We have : [r+ (-s)]a = ra + (-s)a

= ra-sa

Theorem 1.3 (II)

The intersection of any two submodules of a R- module M over a ring R is given a submodule of M.

Proof : Let A and B be any two submodules of a R module M.

Thus A∩B is an additive sub group of M (As A and B are subgroups) and for r ∈ R and a ∈ A∩B => ra∈ A∩B.

Again A and B are submodules.

Thus r ∈ R, a ∈ A => ra ∈ A

Also r ∈ R, a ∈ B => ra ∈ B

Thus ra ∈ A∩B

Hence A∩B is a submodule of M

Theorem 1.3 (III)

Arbitrary intersection of submodules is a submodule.

Proof : Let $\{M_i\}$ for I =1,2,3…. , n be a set of submodules of a module M over the ring R.

Let P = $\{\cap M_i\}$ then we have

(i) P is an additive subgroup of
(ii) For r ∈ R and a ∈ P => r ∈ R and a ∈ Mi for each I = 1,2,3,…, n
Since each Mi is a submodule of M
Hence r ∈ R and a ∈ each Mi => ra ∈ each Mi =>ra ∈ P
Thus P is a submodule.

Theorem 1.3 (iv)

The union of any two submodule of a module is submodule if any only if one is contained in the other.

Proof Let M1 and M2 be any two submodules of a module M over a ring R.

Let M1 ⊆ M2. To prove M1U M2 is a submodule => M1 U M2 is submodule.

Also M1UM2 = M1 if M2 ⊆ M1 and M1 is a submodule => M1 U M2 is submodule.

Conversely: We assume M1 ⊆ M2 is a submodule of module M.

To show that either M1⊆ M2 or M2 ⊆ M1.

For this, if possible, let neither M1 ⊆ M2 nor M2 ⊆ M1

Then if a ∈ M, then a is not in M2 and b ∈ M2 then b is not in M1. But M1 U M2 is a submodule of R-Module.

So (a-b) ∈ M1U M2 => either a-b ∈ M1 or a-b ∈ M2

If a-b ∈ M1 and a ∈ M1, then b = a- (a-b) ∈ M which is a similarity if a-b ∈ M2 then a=b + (a-b) ∈ M2 is again a contradiction. Thus M1 ⊆ M2, or M2 ⊆ M1

Again since:

a, b ∈ M1 U M2 and M1U M2 is submodule.

Hence for r ∈ R, a r ∈ M1 U M2 => either a r ∈ M, or a r ∈ M2

Similarly b r ∈ M1 U M2 => either br ∈ M, or br ∈ M2.

But ar ∈ M2 =>a ∈ M2 and br ∈ M1 =>b ∈ M1 which goes against the supposition that a ∉ M2 and b ∉ to M1. So letting neither M1 ⊆ M2 nor M2 ⊆ M1 we arrived at a contradiction. This either M1 ⊆ M2 or M2 ⊆ M1.

Section 2: Linear and Direct sum of two modules:

2.1 Introduction: We shall see how to find out the linear and the direct sum of any two submodules of a given module.

2.2 In this section we give some of the related definitions which has been used in the theorems which will be given in the next section.

Linear sum of two submodules:

Let M1 and M2 be any two submodules of an R- module. Then the linear sum of M1 and M2 is denoted by writing M1+M2 and we define it as :

M1+M2 = [a+b : a ∈M1, b ∈ M2] clearly M1+M2 is a subset of M

Direct sum of two submodules:

Let M1 and M2 be any two submodules of an R- module M. We say that M is the direct sum of M1 and M2 if each element of M can be uniquely expressed as sum of an element of M1 and element of M2.

We denote it by writing M= M1 … M2.

Also, this special kind of the sum can be extended to n numbers of submodules of M.

2.3. In this section we give theorems:

Theorem 2.3 (i)

The linear sum of any two submodules of an R module M is given a submodule of the R-module M.

Proof : Let M1and M2 be any two submodules of an R module M.

Let a = m11+m21 , b+m12 +m 22 be any two elements of M1+M2 such that m11, m12 ∈M1 and m21, m22 ∈M2.

Now (a-b) = (m11+m21) – (m12+m22) = (m11-m12)+(m21 -m22)

Since M1 is an additive subgroup so m11-m12 ∈M1

Similarly, as M2 is also an additive subgroup so m21-m22 ∈M2

Therefore (m11 – m12) + (m21- m22) ∈M1+M2

Thus M1+M2 is additive subgroup of M

Now for any r ∈R we find that ra=r(m11+m21)= rm11+rm21

Since M1 is a submodule so r ∈R, m11 ∈M1 implies rm11 ∈M1+M2

Also, from (1) ra ∈ M1+M2

Therefore, the linear sum M1+M2 is a submodule of R module M.

Theorem 2.3 (ii)

The necessary and sufficient conditions for a module to be a direct sum of its two submodules M1 and M2 are that

(i) M= M1+M2
(ii) M1 ∩ M2 = {0}

Proof: Let M1 and M2 be any two submodules of an R- module M.

Let M be the direct sum of M1 and M2 that is M= M1⊕M2 then by definition of direct sum every element of M can be uniquely expressed as the sum of element of M1 and the element of M2.

That is if m ∈M then m=m1+m2 for m1 ∈M1 and M2 ∈ m2

Thus M= M1+M2 so the first part is done.

We now come to establish the second part: To show M1∩M2 = {0}.

We prove it by contrary method

For this if possible, let x ∈M1∩M2 such that x ≠ 0

Thus x= x+0 ∈M1+ M2, x∈M1, 0∈M2

And x= 0+x ∈M1+ M2, o∈M1, x∈M2

But then x∈ M implies x can be expressed as the sum of elements of M1 and M2 in two ways , which contradicts the fact that the element of M can be uniquely expressed as the sum of element of M1 and M2. Thus 0 is the only element common to both M1 and M2.

Thus M1∩M2 = {0}

Conversely: Let M = M1 + M2 and M1∩M2 = {0}. To prove M + M1 ⊕M2

For M = M1 + M2

Let x ∈M be an arbitrary implies that x= x1+y1, x1∈M1, y1∈M2 which is sufficient to show that the expression $x= x_1 + y_1$ is unique.

For this, let if possible $x= x_1+ y_1$, $x_1 \in M_1$, $y_1 \in M_2$

and x = x2+y2, x2 ∈M1, y2 ∈M2

Then $x=x_1+ y_1= x_2+y_2$ implies that $x_1-x_2 =y_2-y_1 \in$ M1∩M2

But by assumption M1+M2= {0} .

Thus x1-x2= y2-y1, =0 implies that x1=x2 & y1=y2 implies that x=x1+y1 is unique implies that

M= M1 ⊕M2. That is M is the direct sum of M1 and M2.

Section 3

Homomorphism, Isomorphism, Kernel of Homomorphism

3.1 Introduction – An Isomorphism can also be termed as an indirect equality in algebraic system. In fact if the two system have the same number of elements and behave exactly in the same manner , we can call them equal but at times the idea of equality may look little uncomfortable, specially in the case of infinite sets.

3.2 **Definitions** : Homomorphism of Modules : Let M & N be any two R- Modules .Let f:M→N be a mapping then f is called module homomorphism(or R- homomorphism).

If:

i. f(m_1+ m_2) = $f(m_1)$ + $f(m_2)$ for all m_1,m_2 ϵM

ii. $f(rm) = r.f(m)\ \forall\ r\epsilon R$ *and* $m\epsilon M$ Here N is called the homomorphic image of M under f Isomorphism Of Modules : Let M & N be any two R- Modules .Let f:M→N be a homomorphism then f is called isomorphism.

If f is one – one onto also.

Note 1 :- If f:M→N is homomorphism then it's easy to verify that

i. $f(0) = 0$

ii. ii. $f(-m) = -f(m)\ \forall\ m\epsilon M$

iii. $f(m_1 - m_2) = f(m1) - f(m2), \forall\ m1,m2\ \epsilon M$

Kernel of homomorphism :

Let M an N be any two R-modules and f:M→N is homomorphism then kernel of f is denoted by Ker(f) or simply by K(f)is defined as the set of all those elements of M which image is 0 (That is Identity element of N) That is K(f) = {$m\epsilon M$:$f(m)$ = 0}

3.3 Theorem :

Theorem 3.3(i) The Kernel of a homomorphism is a submodule Proof : Let M an N be any two R-modules and f:M→N is homomorphism from M into N Then K(f) = {$m\epsilon M$:$f(m)$ = 0} Since $f(O)$ = 0 atleast $O\epsilon K(f)$ => $K(f)$is non empty .In other words $K(f)$ is non empty subset of M .

Also if $m1, m2 \in K(f) then\ f(m_1) = 0,\ f(m2) = 0\ and\ f(m1 - m2) = f(m_1) - f(m_2) = 0 - 0 = 0$
$\Rightarrow m1 - m2 \in K(f)$

Hence $K(f)$ is an additive subgroup of M.

Again Let r$\in R$, $m \in K(f)$ such that $f(m) = 0$ Then $f(rm) = rf(m) = r.0 = 0 \in K(f)$ Thus $K(f)$ is submodule of M

Theorem 3.3(ii): A Module homomorphism is f is isomorphism only if and only if K(f) = 0
Proof : Let f:M→N be homomorphism of an R-modules M onto an R-module N Let $f(f) = 0$
To prove f is an isomorphism. For this it is sufficient to show that f is one- one.

For this, let $m_1, m_2 \in M$ such that :

$f(m_1) = f(m_2) \Rightarrow f(m_1) - f(m_2) = 0 \Rightarrow f(m_1 - m_2) = 0 \Rightarrow m_1 - m_2 \in K(f) \Rightarrow m1 - m2 = 0 \Rightarrow$
$m1 = m2$.

That is $f(m_1) = f(m_2) \Rightarrow m_1 - m_2 = 0 \Rightarrow$ m1 = m2) f is one –one That f is homomorphism and also one –one so f is isomorphism

Conversely: Let f is isomorphism. To show that $K(f) = 0$.

For, f is isomorphism implies that f is one-one. For $m \in K(f)$ then $Ker(f) = f(m) = 0 = f(O)$
$\Rightarrow m = 0 \Rightarrow K(f) = 0$

Theorem 3.3. (iii): The range of module homomorphism is sub module

Proof: Let f:M→N be an homomorphism of an R- Modules into an R- Module N.

Let $I(f)$ is range of f Then $I(f) = \{ f(m) : m \in M\}$.

To prove that $I(f)$ is submodule. For this we establish two conditions.

For let $f(m_1)$ and $f(m_2)$ be members of $I(f)$ then $f(m_1) - f(m_2) = f(m_1 - m_2) \in I(f)$

(since $(m1 - m2) \in M$].

Thus $I(f)$ is an additive subgroup of N Also let $r \in R$ and $f(m) \in I(f)$ then $rf(m) = f(rm) \in I(f)$ as $rm \in M$ (Since M is R- Module). Thus $I(f)$ satisfies the conditions to be submodule of N Hence $I(f)$ is submodule of N

Section 4.

4.1 **Quotient Modules:** The study of Quotient Modules is closed to the study of normal subgroup of a group.

4.2 **Definitions:** Let A be submodule of an R- Modules M. Then by definition M is an additive group and A is its subgroup. $A + m\ for\ m \in M$ is a caset of A in M Let M/A = { $A + m\ for\ m \in M$} is the set of all casets in A in M Then M/A is called the quotient modules of M with respect to submodule A where composition is defined by

(A+m1) + (A+m2) = A+(m1+ m2), ∀ $r \epsilon R$ and m is in M

4.3 Theorems:

4.3 (i) Fundamental theorem on homomorphism of modules: Let f: M

→ N be a homomorphism of an R-module M onto a R-module N with K(f) = A then N is isomorphic to M/A That is N is equivalent to M/A

Proof: In the light of the hypothesis we know that K(f) is a submodule of M. But by question K(f)=A ⇒A is a submodule of M ⇒M/A is a quotient module.

Thus M/A = {m +A : mϵA}

Also, for every m1ϵA, f(m1)=0.

Also, range of f = I(f) = N(as f is onto ⇒ I(f) = co-domain = N).

We now define a mapping Ψ: N → M/A given by Ψ(f(m)) = m+A , f(m) ϵ N , mϵM.

Thus Ψ is well defined.

Further we take m_1 m_2 ϵ M so that f(m_1), f(m_2) ϵN.

Also, Ψ(f(m_1)) = m_1+A and Ψ(f(m_2))=m_2+A

Then Ψ(f(m_1)) = Ψ(f(m_2)) ⇒ m_1+A = m_2+A ⇒ m_1 - m_2 ϵA ⇒ f(m1 - m2) = 0 ⇒ f(m1) - f(m2) = 0 ⇒ f(m1) = f(m2)

That is Ψ(f(m_1)) = Ψ(f(m_2)) ⇒ Ψ(f(m_1+m_2)) = (m_1+m_2) + A = (m_1+A) + (m_2+A) = Ψ(f(m1)) + Ψ(f(m2)).

Thus Ψ is one -one.

Further, Let $r\epsilon R$ then $\Psi(r\ f(m)) = \Psi(f(rm)) = rm + A = r\ (m+A) = r\ \Psi(f(m))$.

So, Ψ satisfies all the conditions to be an isomorphism.

Thus M/A is an isomorph image of N That is N is equivalent to the quotient module of M with respect to the submodule A of M.

Theorem (4.3(ii))

Let A and B be any two submodules of R-module M then A+B/B is equivalent to A/A∩B.

Proof: By question A and B are any tow sub modules of R-module M.

Also, $b = 0+b\ \epsilon A+B$ as $0\epsilon A$, $b\ \epsilon B$

Thus $B \subseteq A+B$ and $A\cap B \subseteq A$ so B is a submodule of A+B

Similarly A∩B is a submodule of A.

Thus the quotient modules (A+B)/B and A/A∩B both exists.

We now define a mapping $\Psi: A \rightarrow (A+B)/B$ given as $\Psi(a) = a+B$, for every $a\epsilon A$………………..1

Let $x,y\ \epsilon A$ and $x=y \Rightarrow x+B = y+B \Rightarrow \Psi(x) = \Psi(y) \Rightarrow \Psi$ is well defined.

Again, Let $x,y\ \epsilon\ A$, then $\Psi(x+y) = x+y+B = (x+B)+ (y+B) = \Psi(x) + \Psi(y)$.

Also for any $r\ \epsilon R$, $x\epsilon A$, $\Psi(rx) = rx+B = r(x+B) = r\ \Psi(x)$

Thus Ψ is a homomorphism.

Also, let $z\epsilon\ (A+B)/B \Rightarrow z= x+y+B$ for some $x\epsilon A$, $y\epsilon B \Rightarrow z = r+B$ as $y\epsilon B \Rightarrow y+B=B$

But $x\epsilon A$, then $\Psi(x) = x+B$ so that $\Psi(x) = z$

Therefore Ψ is onto.

Finally, $\because K(\Psi) = \{x\epsilon A, \Psi(x) = B\} = \{x\epsilon A: x+B=B\} = \{ x\epsilon A: x\epsilon B\} = A\cap B$

Then $K(\Psi) = A\cap B$

Thus Ψ is onto homomorphism from A to (A+B)/B with $K(\Psi) = A\cap B$ Hence by the first theorem of isomorphism, (A+B)/B is isomorphic to A/A∩B That is (A+B)/B is equivalent to A/A∩B

5. Simple and Semi-Simple modules

5.1 **Introduction:** Simple and Semi-simple modules are in-fact an advanced topic of module

5.2 **Definitions:** Simple module: An R-module M is said to be simple if it has no submodule other than itself and the zero submodule. Semi simple module: An R-module m is said to be semi-simple if it can be expressed as a direct sum of simple submodules.

Example 1: An abelian group G≠0 is a simple z- module if and only if it cyclic of prime order

5.3 Theorems: 5.3(i) Schur's Theorem: If m is a simple R-module and N is any R-module then:

(i) Every non-zero homomorphism f: M → N is one-one (Injective)

(ii) Every non-zero homomorphism f: M → N is onto (epimorphism or surjective)

(iii) Endg (M) is a division ring, where $EndR(M)$ = Hom_g (M,M)

Proof: By hypothesis f: M → N is homomorphism ⇒ K(f) (Ker(f)) is a submodule of M

(i) M is simple ⇒ either K(f) = [0] or K(f) = M
But f≠0 then K(f) ≠M ⇒ K(f) = [0]

Then by a theorem f is one-one

(ii) Since f:M→N is a homomorphism and f≠0
Thus I(f) = range of f=M ⇒ f is onto (or surjective)

(iii) By question M is simple. If f≠0 and f: M → M then due to (i) and (ii) f is automorphism. Then f is a unit in the ring Endg (M).

Thus Endg (M) is a division ring

Theorem 5.3 (ii) Let M be an R-module then following conditions are equivalent

(i) M is sum of simple submodules
(ii) M is a semi-simple module
(iii) Every submodule of M is a direct sum of M

Proof: Since (i) implies (ii)

For let {Mi : i ϵ I} be a family of simple submodules of M

If M=$\sum_{i\epsilon I} Mi$ then by a theorem with N=0, there is a subset I' of I such that M= ⊕ Mi,

for i ϵ I'

⇒M is the sum of simple submodules

(ii) ⇒(iii) Let N be a submodule of M and let M= ⊕ Mi, for i ϵ I', M= ⊕ Mi, for i ϵ I

Then by a theorem there is a subset I' of I such that M⊕Mi⊕N, i ϵI'.

Let a ∈ M, N = Ra then N is a finitely generated R module so N has a maximal submodule L.

Also, L has a complementary submodule L' such that M = L ⊕ L`

Thus N= L ⊕ (NUL`) implies that NUL` is a submodule of M which is isomorphic to N/L . This implies that NUL` is simple.

Also, every nonzero submodule contains a simple submodule because every nonzero sub module of M contains the cyclic submodule.

Let P= the sum of simple submodule of M implies that M = P ⊕ P`, P` is some submodules.

Also, if p` ≠0 then p` contains simple submodules which is not possible as all of the simple submodules are contained in P.

Thus M= P implies that M is the sum of simple submodules.

Theorem : 5.3 (iii)

Every submodule and every homomorphic image of a semi module is semi module.

Proof :

Let M be a semi simple R-module. N be a submodule of M and L be a submodule of N.

Then as we know that there will exist a submodule L` of M.

Such that : M = L⊕ L`

In this case L ⊕ (L`∩ N) =N

Thus by - **Theorem 5.3 (iii),** every submodule of N is direct sum of N. Thus N is simple.

Also we know that every homomorphic image of R- module M is isomorphic to M/N for some sub module N of M.

But M/N ≅N`.

Thus, M is semi simple.

Section – 6

Free module

6.1 Introduction: Free module is also an advance topic of modules.

6.2 Definitions:

Finitely Generated Module:

A R- module M is called finitely generated if there exist finite number of elements.

m 1 m2,…..mn in M such that every m in M can be represented as m = $\sum_{i=1}^{n} rimi$ for same ri ∈ R

Basis for a module: Let M be a module over a ring R with unity and let S be a subset of M then M then we say that S is the basis for M if the following two conditions are satisfied:

(i) S generates M
(ii) S is linearly independent

Free module: Let M be a R- module then we say that M is a free module if we get a subset S of M such that:

(i) S generates M
(ii) S is a linearly independent set

Example : If M = (0), then the set Φ Is its basis. So, it is free module.

6.3 Theorems:

Theorem 6.3(i)

Let

(i) Let R be a ring and M be a module over R
(ii) S be a nonempty set
(iii) {xi : i∈S } be a basis of M
(iv) N be an R-module and {yi : i ∈ S } be a family of element of N.

Then there exists a unique homomorphism f: M → N such that f(xi) =yi, for every i in S.

Proof : Since {xi : i∈S } is a basis for M

Thus every element of M can be expressed uniquely as linear combination of elements of {xi : i∈S }

If x∈M then we can get an unique family {ri : i∈S } of elements of R such that :

$x=r_1x_1 +r_2x_2+….+r_kx_k = \sum_{i\in S} rixi$ ……………………………………………….. 1

Let f: M→N such that f (x) = $\sum_{i\in s} riyi$, for all x∈M , for all yi ∈ N…………………………………… 2

Thus, f is a homomorphism.

Also f(x) = $\sum_{i\in s} rif(xi)$ ……………………………………..3

Thus f is unique also, from 2 and 3 f(xi) = yi, for every i∈ S

Theorem 6.3 (ii)

Every finitely generated module is a homomorphic image of a finitely generated free module.

Proof : Let the set { x1, x2,…, xn } generates a R- module M.

Let e_i = { o, o, …., 1,0…0 }.

Clearly the set { e1,e2,…..en} is linearly independent over R and generates a free module $R^{n.}$

Now let a mapping Ψ: $R^{n.}$ → M defined by Ψ (x) = $\sum_{i=1}^{n} rixi$, x ∈ R^n so that x = $\sum_{i=1}^{n} riei$,

Thus Ψ is well defined [As every element of R^n has a unique representation as $\sum_{i=1}^{n} riei$]

Further let x =$\sum_{i=1}^{n} riei$ and y= $\sum_{i=1}^{n} siei$ can be two elements of $R^{n.}$

Then Ψ (x+y) = Ψ ($\sum_{i=1}^{n} riei + \sum_{i=1}^{n} siei$) = Ψ $\sum_{i=1}^{n} riei\ (\sum_{i=1}^{n} riei + \sum_{i=1}^{n} siei) = \Psi$ $\sum_{i=1}^{n}(ri+si)ei) = \sum_{i=1}^{n}(ri+si)\ xi$ = $\sum_{i=1}^{n} rixi + \sum_{i=1}^{n} sixi$ = Ψ (x) + Ψ (y)

That is , Ψ (x+y) = Ψ(x) + Ψ(y)……………………………….1

Also, Ψ(rx) = Ψ ($\sum_{i=1}^{n} rriei$) = $\sum_{i=1}^{n} rirxi$ = r $\sum_{i=1}^{n} rixi$ = r Φ (x)…………………….2

Thus 1 and 2 allow us to speak that Ψ : $R^{n.}$→ M is a homomorphism.

Theorem 6.3 (iii)

Let: (I) R be a ring and M be a free module with basis B.

(II) N be any R-module and f:B → N be any mapping

Then there exists a unique R- module homomorphism

Ψ : M→ N

Proof : Let B = { xi : i ∈ I } be a basis of M.

Then as we know that if x ∈ M then x can be expressed as

x= Ψ $\sum_{i\in \mathrm{I}} rixi$, where ri is in R.

Let Ψ: M→ N be a mapping given by Ψ (n)= $\sum_{i\in \mathrm{I}} rif(xi)$

Then for x1y ∈ M, ri ∈ R, si ∈ R we have

x= $\sum_{i\in \mathrm{I}} rixi$, y = $\sum_{i\in \mathrm{I}} sixi$ then we see that

Ψ (x+y) = Ψ [= Ψ = $\sum_{i\in \mathrm{I}} rixi$ + $\sum_{i\in \mathrm{I}} sixi$] = Ψ [$\sum_{i\in \mathrm{I}}(ri + si)xi$] = $\sum_{i\in \mathrm{I}}(ri + si)f(xi)$ = $\sum_{i\in \mathrm{I}} rif(xi)$+$\sum_{i\in \mathrm{I}} sif(xi)$ =

= $\sum_{i\in \mathrm{I}} rif(xi)$ + $\sum_{i\in \mathrm{I}} sif\ (xi)$= Ψ (x) + Ψ (y)

That is:

Ψ (x+y) = Ψ (x) + Ψ (y)……………………………….1

Also, we see that for r ∈ R, x ∈ M,

Ψ (rx) = $\sum_{i\in \mathrm{I}} rif(xi)$ = r $\sum_{i\in \mathrm{I}} rif(xi)$ = r Φ(x)………………………..2

Thus from 1 and 2

Ψ is R- module homomorphism

Section 7

Noetherian and Atinian Modules; ascending chain condition, descending chain condition.

7.1 Introduction: In fact this very topic comes within the advance study of modules.

7.2 Definitions:

Modules with chain conditions:

(i) Let M be the module over the ring R
(ii) S = {Mα : $\alpha \in$ I} = the set of all submodules of M

Then we speak that S satisfies the conditions given below:

(a) The ascending chain condition: If every increasing chain of submodules $M1 \subseteq M2 \subseteq \ldots$.. in S is stationary.

That is there exists a positive integer n such that $Mn = M_{n+1} = M_{n+2} = \ldots\ldots$

(b) The descending chain condition: If every decreasing chain of submodules $M1 \subseteq M2 \subseteq M3 \subseteq$ in S is stationary.

That is there exists a positive integer n such that $Mn = M_{n+1} \ldots\ldots\ldots$..

<u>**Noetherian module:**</u>

If a R-module satisfies the ascending chain condition.

<u>**Artinian module:**</u>

If a R-module satisfies the descending chain condition.

7.3 Theorems

Theorem 7.3 (i) Every submodule of a Noetherian (Artinian) module is Noetherian (artinian).

Proof: Let N be a submodule of a Noetherian (Artinian) module M
To prove that N is Noetherian (artinian)

For this, since given M is Noetherian (artinian) implies M satisfies ascending (descending) chain condition of submodules.

We also know by a theorem that every nonempty set of submodules of modules M has a maximal (minimal) element. This Implies that N is Noetherian (artinian).

Therefore, every submodule N of Noetherian (artinian) module M is Noetherian (artinian).

Theorem 7.3 (ii):

Every homomorphic image of a Noetherian (artinian) module is Noetherian (artinian).

Proof : Let M be a Noetherian (artinian) module. Let also assume that f be module homomorphism of M.
To show f(M) is Noetherian (Artinian)

For this, since M is Noetherian so we have an ascending chain condition so if
S ={ Mi : i ∈ I} is the set of submodules of M then M1 ⊆M2⊆M3⊆……⊆Mn⊆.... S.t $M_k = M_{k+1} = \ldots$

for some positive integer K.

We also have a descending chain condition for the set S = {N_i : i ∈ I } of submodules of M as

N1 ⊆ N2 ⊆ N3⊆ ….⊆ Nn ⊆… s.t Nr= $N_{r+1} = \ldots\ldots$

For some positive integer r.

Now, in case of Noetherian

f(M1) ⊆f(M2) ⊆…..f (Mn) ⊆… s.t f (Mk) = f(M_{k+1}) =……

Similarly for the case artinian:

F(N1) … f(N2) ……. F(Nn)…..st f (Nr) = f (Nr+1)=……

Implies f (M) is Noetherian (artinian).

Thus, every homomorphic image of a Noetherian (artinian) module is Noetherian (artinian).

Theorem 7.3 (iii) :

Let N be a submodule of a R-module M

Then M is Noetherian (artinian) if and only if both N and M/N are Noetherain (artinian)

Proof : Firstly we assume that N and M/N are both Noetherian.
Also, let L be a submodule of M, then (L+N)/N is a submodule of M/N
Since M/N is Noetherian implies that (L+N)/N is finitely generated.
Thus we get a, a2 …, n L/ N∩L such that L/N∩L =($\bar{a_1}$, $\bar{a_2}$, …, $\bar{a_n}$)
So L = (a1,a2,….., an) +(N∩L), for ai ∈L

Again N is Noetherian , N∩L is a submodule of N implies that N∩L is finitely generated.
But then there exists b1, b2….bm…. N∩L such that N∩L = {b1,b2,….,bm}
So L = {a1, a2,… an} + { b1, b2,…, bm} implies that L is finitely generated and L is any submodule of M which means M is Noetherian.

<u>Conversely</u>

We suppose M to be Noetherian . To prove M/N is Noetherian. But M is Noetherian implies that every submodule is Noetherian implies that N is Noetherian.

Let f:M → M/N be a canonical homomorphism and let $\bar{M1}$ C $\bar{M2}$ C…. be an ascending chain of submodule of M/N.
If Mi = f^{-1} ($\bar{M}$i) , then we have Ma C M2CM3C…. is an ascending chain of submodules of M.
Since, M is Noetherian, so there exists a positive integer r such that $M_r = M_{r+1}$ = …….

Then , we have f (M_i) = $\bar{M_i}$ for all i ≥r implies thar $\bar{M_r}$ = $\bar{M}_{r+1}$ = …….

Implies that M/N is Noetherian.

End.

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