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PART - I

SUBJECT
DISCRETE MATHEMATICS

PAPER - III

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SET, RELATIONS AND FUNCTIONS

The Algebraic Laws of Sets

Cartesian Product of Two Sets

Function or Mapping

Relation

The Algebraic laws of Sets :

(i) Idempotent laws : If A be any set, then

$$A \cup A = A \text{ and } A \cap A = A$$

(ii) Commutative laws : If A and B be two sets,

Then $A \cup B = B \cup A$

$$A \cap B = B \cap A$$

(iii) Associative laws : If A, B and C be any three sets, then

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

(iv) Distributive laws : If A, B and C be three sets then

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(v) De Morgan's Laws : if A and B be any two sets, then

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Where A´ and B´ are complements of A and B respectively

If A, B and C be any three sets then

$$A - (B \cup C) = (A - B) \cap (A - C)$$

$$A - (B \cap C) = (A - B) \cup (A - C)$$

Cartesian Product of two sets :

If A and B are any two non-empty sets, then the cartesion product of A and B is denoted by A × B and it is the set of all ordered Pairs (a, b) where a   A and b   B

i.e. A×B={(a, b) /a   A, b   B}

Exampls :

A = {1, 2, 3}, B ={a, b} then

A × B = {1, 2, 3} × {a, b} = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)

B × A = {a, b} × {1, 2, 3}

{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)}

A × A = {1, 2, 3} × {1, 2, 3}

= {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3) (3, 1), (3, 2), (3, 3)}

B×B = (a, b)×(a, b)

= {(a, a), (a, b), (b, a), (b, b)}

Note :–

(i) In general A × B   B × A

(ii) If A or B is a null set then A×B=

(ii) If n(A) = p, n(B) = q

then n(A×B) = pq

Questions :

Find x and y if

(3x + y, x–1) = (x + 3, 2y – 2x)

Soln. The ordered pairs are equal if

3x + y = x + 3 and x – 1 = 2y – 2x

if 3x–x + y = 3 or, x + 2x = 2y + 1

2x + y = 2 or, 3x – 2y = 1 --------(ii)

2x = 3 – y

$x \quad \frac{3-y}{2}$ -------------(i)

Putting the value of x in equation (ii) we get

3x – 2y = 1 from (i)

or, $3 \quad \frac{3-y}{2} \quad -2y \quad 1$ $\qquad x \quad \frac{3-y}{2}$

or, $\frac{9-3y-4y}{2}=1$ $\qquad x \quad \frac{3-1}{2}$

or, 9 –7y = 2 $\qquad x \quad \frac{2}{2} \quad 1$

or, 9 – 2 = 7y

7 = 7y

y = 1

After solving $\boxed{x \quad 1, y \quad 1}$

Questions

If A = {1, 2, 3}, B = {2, 4, 5} find

(i) $(A \cap B) \times (A – B)$

(ii) A × (A–B)

(iii) $(A \quad B) \times (A \cap B)$

Solutions :

(i) $(A \cap B) = \{1, 2, 3) \cap (2,4,5)$

$A \cap B = \{2\}$

$A – B = \{1, 2, 3) – \{2, 4, 5\}$

$A – B = \{1, 3\}$

$(A \cap B) \times (A – B) = \{2\} \times \{1, 3\} \quad \{(2,1),(2,3)\}$

(ii) $A \times (A – B) = \{1, 2, 3\} \times \{1, 3\}$

$= \{(1, 1), (1, 3), (2, 1), (2, 3) (3, 1), (3, 3)\}$

(iii) $A \quad B = (A – B) \cup (B – A)$

$B – A = \{2, 4, 5\} – \{1, 2, 3\}$

$B – A = \{4, 5\}$

$(A \quad B) = (A – B) \cup (B – A)$

$= \{1, 3\} \cup \{4, 5\}$

$A \quad B = \{1, 3, 4, 5\}$

$(A \quad B) \times (A \cap B) = \{1, 3, 4, 5\} \times \{2\}$

$= \{(1, 2), (3, 2), (4, 2), (5, 2)\}$

4.6.4 If $A = \{x | x \quad N \text{ and } x \quad 3\}$

$B = \{x | x^2 - 16 \quad 0 \text{ and } x \quad 0\}$

Find B × A where N is a set of natural number.

Solutions :

$A = \{1, 2\}$ $\qquad x^2 - 16 \quad 0$

$b = \{–4\}$ $\qquad x^2 \quad 16 \quad x < 0$, i.e. $x = –4$

$B \times A = (–4) \times (1, 2)$

$= \{(–4, 1), (–4, 2)\}$

4.6.5 If $A \quad \{x \mid x$ is a positive prime. There are no negative prime < 8|

B = (6, 7, 8), C = (7, 8, 9)

Then find $(A \cap B) \times (B \cap C)$

A prime number is a natural number other than one whose only factors are one and it self

Solution :

A = {2, 3, 5, 7}, B = {6, 7, 8}, C = {7, 8, 9}

$A \cap B = \{7\}$ and $B \cap C = \{7, 8\}$

$(A \cap B) \times (B \cap C) = \{7\} \times \{7, 8\} = \{(7, 7), (7, 8)\}$

4.6.6 If $A = \{x \mid x^2 - 5x \quad 6 \quad 0)$, $B = \{2, 4\}$, $C = \{4, 5\}$

then find (A–B) × (B–C)

Solutions :

$A = x^2 - 5x + 6 = 0$

$A = x^2 - 3x - 2x + 6 = 0$

or, $x(x-3)-2(x-3) = 0$

or, $(x-2)(x-3) = 0$

$x - 2 = 0 \qquad x = 2$

$x - 3 = 0 \qquad x = 3$

A = {2, 3}, B = {2, 4}, C = {4, 5}

A – B = {3}, B – C = (2)

Therefore (A–B) × (B–C) = {3} × {2} = {3, 2}

4.6.7 If A = {0, 3}, B = {4, 5}, C = {7, 12}

D = {13, 5}

Verify that (A × B) $\cap$ (C×D) $(A \cap C) \times (B \cap D)$

Solutions :

A × B = {0, 3} × {4, 5}

A × B = {(0, 4), (0, 5), (3, 4), (3, 5)}

C × D = {7×12} × {13, 5}

= {(7, 13), (7, 5), (12, 13), (12, 5)}

$A \cap C = \{0, 3\} \cap \{7, 12\}$

$B \cap D = \{4, 5\} \cap \{13, 5\} \quad \{5\}$

L.H.S. = (A × B) $\cap$ (C × D)

= {(0, 4), (0, 5), (3, 4), (3, 5)} $\cap$ {(7, 13), (7, 5), (12, 13), (12, 5)}

R.H.S $(A \cap C) \times (B \cap D)$

$\{5\}$

Hence L.H.S = R.H.S.

FUNCTIONS OR MAPPINGS

4.7.1 Let X and Y be two non-empty sets. By given some rule, if each element x corresponds to a unique element y Y, then that rule is called a map of x into y and it is denoted by *f*.

The symbol $f : X \quad Y$

is sometimes used as an abbreviation or “f is a function from X to Y”

or “*f* maps the set X into the set Y”

If x X, then the element of y which corresponds to x is denoted by *f*(*x*) and *f*(*x*) is called the image of *x* in y under the map f.

In the map *f* : X Y, *x* is called the domain of the map and the subset of the elements of y which are image of some element of x is called the range of the map.

The mapping or function f is said to be well defined if.

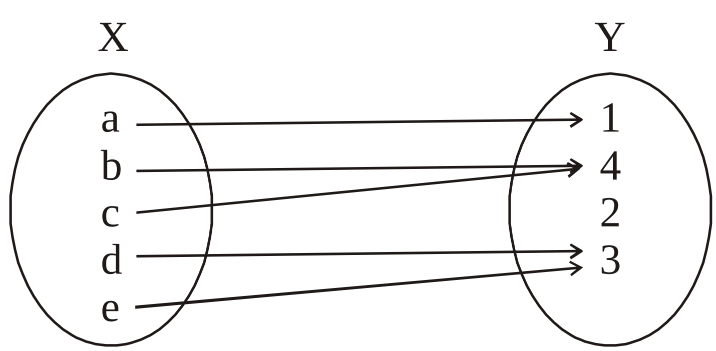
(i) Every element *x* X has an Image *f*(*x*) in *y*.

(ii) An element *x* X has only one image *f*(*x*) in y. *i.e.* Two or more elements of y cannot be the image of the same element of X.

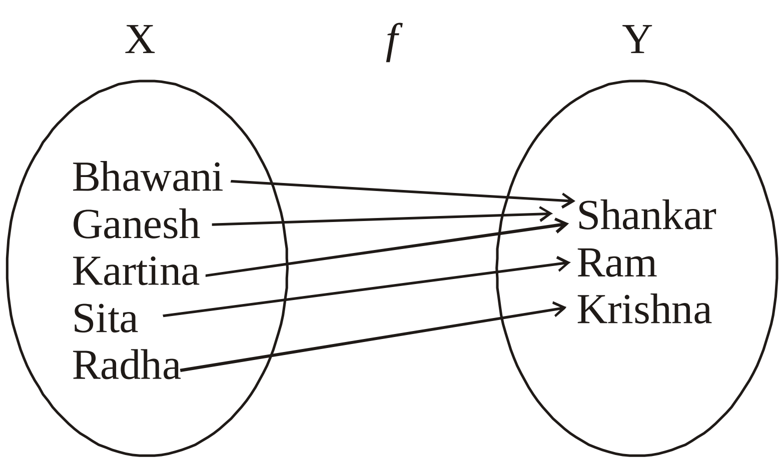
But it is possible that two or more elements of X may have the same Image of Y.

There may be also some element in y which may not be the Image of any element in X.

Examples (i)

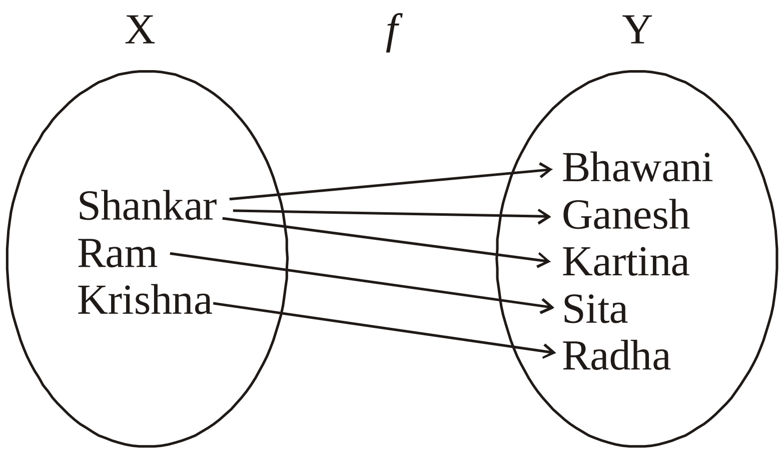


Example (2) This correspondence describes a mappings, because it is satisfies our intuitive concept required of a correspondence to be called mapping



The correspondence shown above also describe a mapping of set X into the set Y.

Example (3) The following correspondence <u>does not describe a mapping of the of</u> the set X into the set Y.



This is so because an elements Shanker in X is associated with three differents elements Bhawni, Ganesh and Kartika in Y.

4.7.2 Types of Mappings : Let $f : x \quad y$

(i) The mapping f is said to be one–one if different elements in × have different f – Images in y.

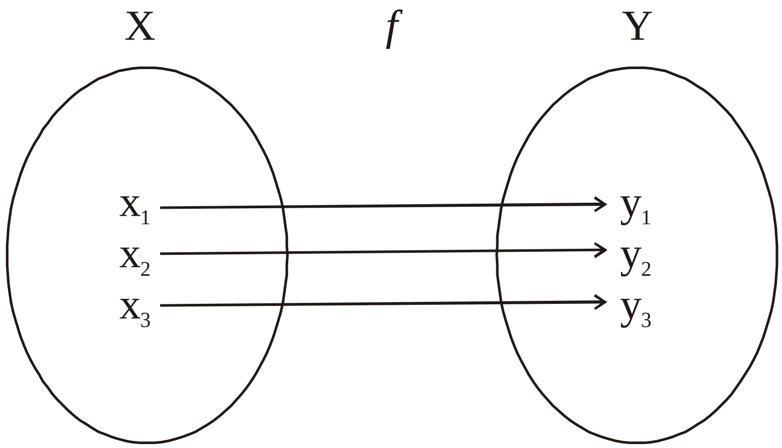
i.e. if $x_1 \quad x_2, x_1, x_2 \quad X$

or, $f(x_1) \quad f(x_2)$

or, Equivalently $f(x_1) \quad f(x_2) \quad x_1, x_2 \quad X$

$x_1 \quad x_2$

Example :

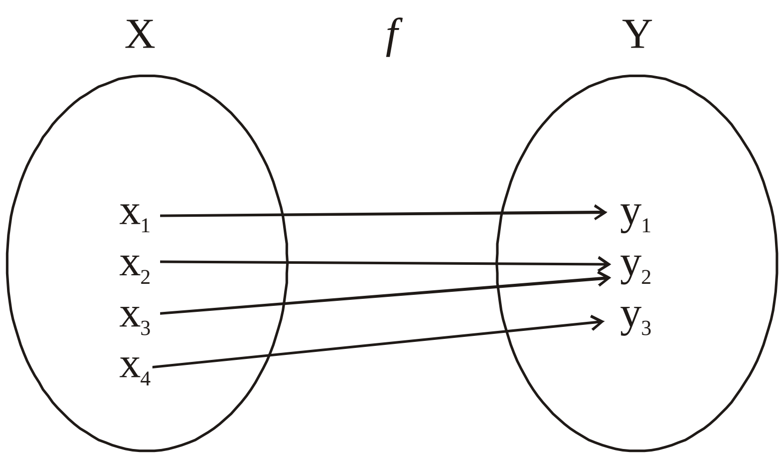


(ii) The mapping f is said to many one if two or more different elements in X have the same *f*–Image in Y.

i.e if $f(x_1) = f(x_2);\ x_1, x_2 \quad X \quad x_1 \quad x_2$

Example :

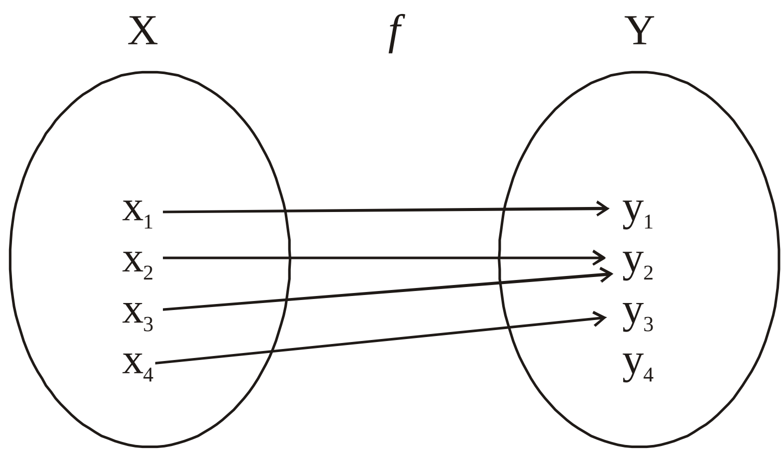
Example of many one mapping



(iii) The mapping *f* is said to be into if there is at least one element in y which is not the *f*–Image of any element in X.

In such mapping *f*–Image of X is a Proper subset of Y.

Example :

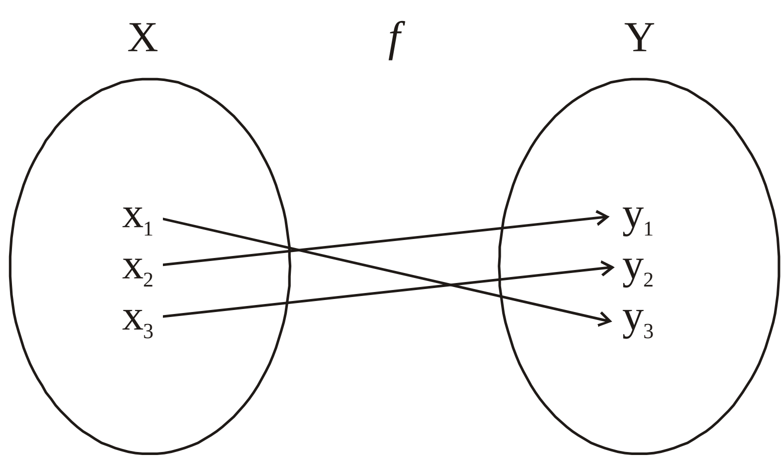


<u>Example of into mapping</u>

(iv) The mapping is said to be into if every element in y is the *f*–Image of at least one-element in X.

In such mapping *f*–Image of x is equal to y.

i.e. $\{f(x) = y,\ x \quad X$



Example of onto mapping

Hence mapping can be of the following four types only.

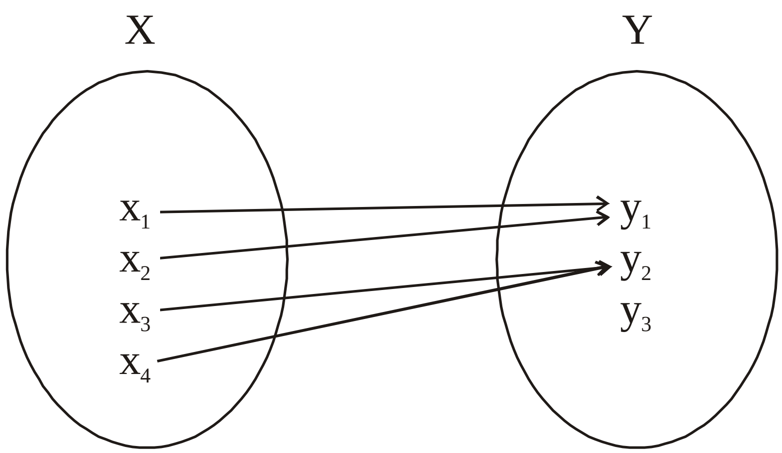
(i) Many – One into mapping

(ii) Many – One onto mapping

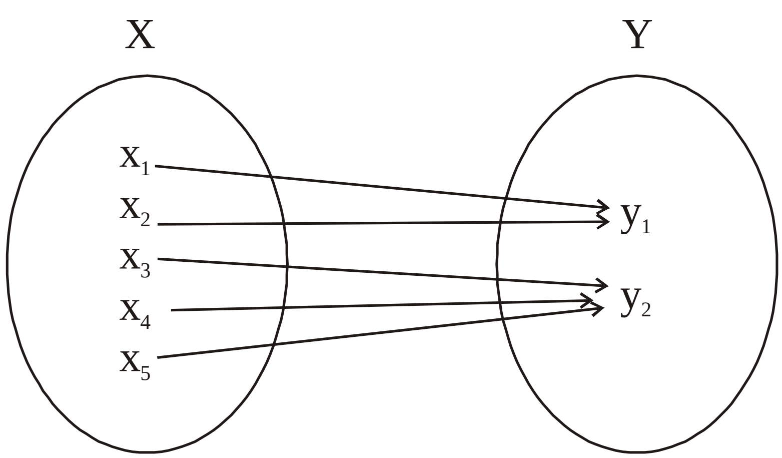
(iii) One – One into mapping

(iv) One – One onto mapping

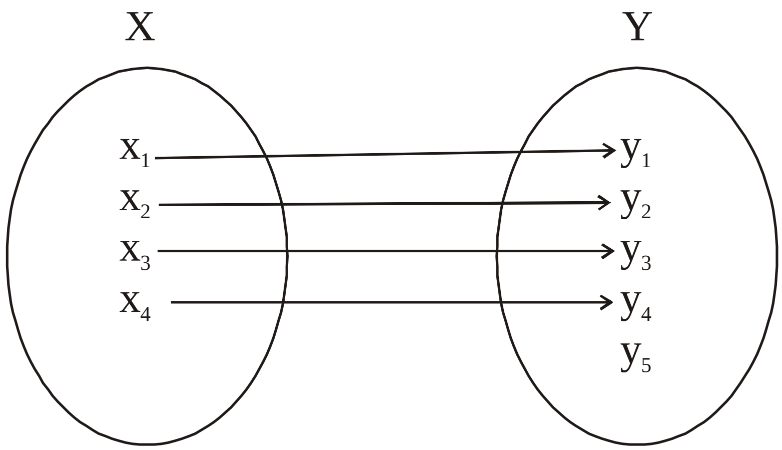
The following diagrams will make mapping more clear.



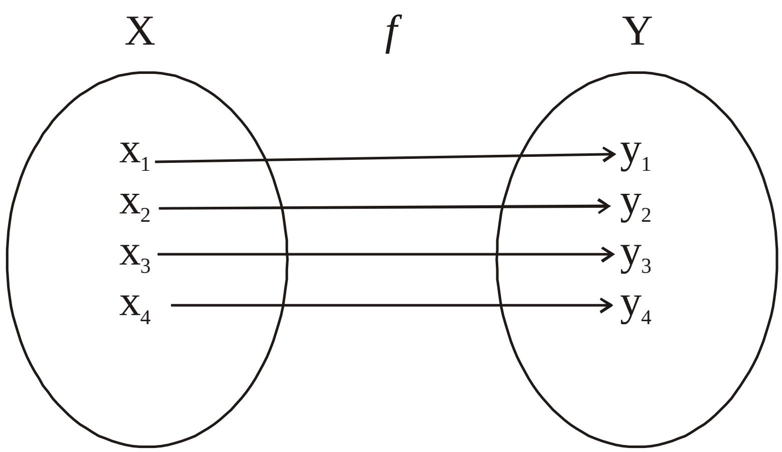
Many–one into mapping



Many – one onto mapping



One-One into mapping



One-One onto mapping

Note :

An onto mapping is also called a surjective map.

An one-one into mapping is also known as Injective map.

A one-one onto mapping is also known as Bijective map.

Constant Function :

If every element of the domain is mapped to unique element of the domain or the domain consists of only one element.

Identity Function :

If $f(x) \quad x — x \quad \text{B}$ in this case $\text{A} \quad \text{B}$

Sometimes it is defined as $f : \text{A} \quad \text{A}$ and $f(x) \quad x, —x \quad \text{A}$

For Example :

(a) Let $f : \text{R}^{+} \quad \text{R}$ be a function defined as $f(x) = 2(x + 2)$ clearly f is 1 – 1

because if $2(x + 2) = 2(y + 2)$ then

$2x + 4 = 2y + 4$

$2x = 2y$ This f is one-one function

$x = y$

(1) Define $f : \text{R}^{+} \quad \text{R}^{+}$ by $f(x) \quad e^{x}, \forall x \quad \text{R}$ clearly f is one-one function.

because if $f(x_1) \quad f(x_2)$ then

$e^{x1} \quad e^{x2}$

$e^{x1-x2} \quad 1$

$x_1 - x_2 \quad 0$

$x_1 \quad x_2$

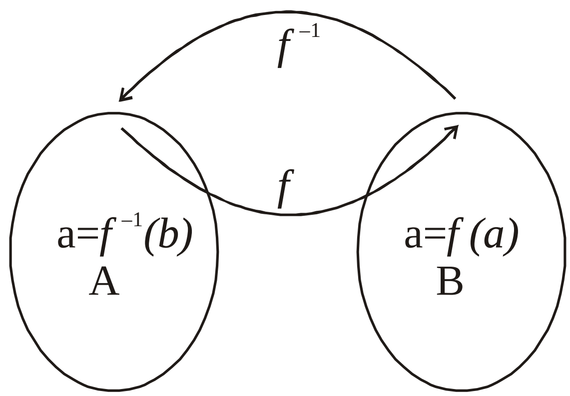
(c) Let A = {5, 6, 7) and B {a, b}. Then the Mapping $f : \text{A} \quad \text{B}$ is defined as $f(5) =$ a, $f(6) =$ b, $f(7) \quad a$

clearly f is not one – one

but f onto function.

Inverse Functions and Operations on Function :

Let f be a bijective function from the set A to the set B. The Inverse function of *f* is the function that assigns to an element b B The unique element a in A such that $f(a)$ b. The inverse function of *f* is denoted by f^{-1}. Hence $f^{-1}(b) \quad a$ when $f(a) \quad b$



The function f^{-1} is the inverse function of f.

Note : Abijective function is called invertible since we can define an inverse of this function.

Product or Composite Mapping

Let g be a function from the set A to the set B and let *f* be a function from the set B to the set C. The composition of the functions *f* and g denoted by (fog) is given by

$$(fog)(x) \quad f(g(x), \forall x \quad A$$

For Example :

The composite function f and g denoted by fog.

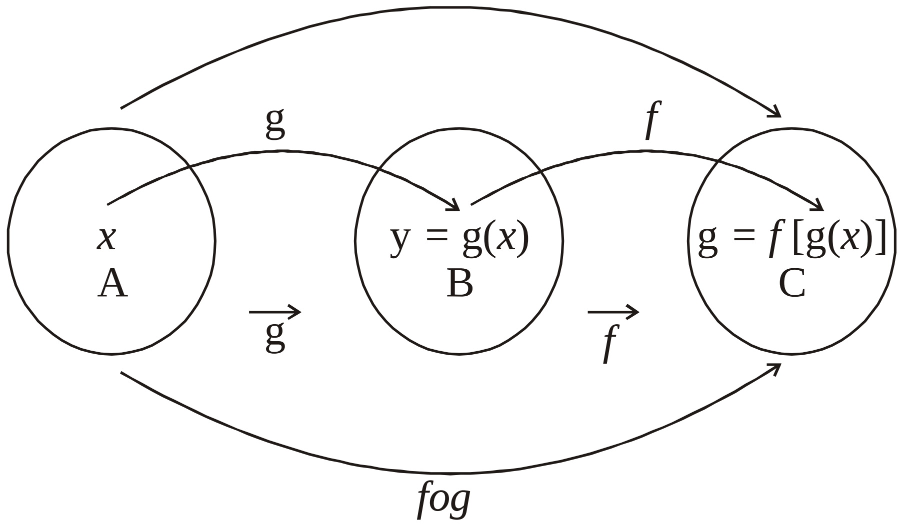


Figure shows the composite function of *f* and *g*

Example (1)

Let f : Z be a function defined by

$f(x) = 2x + 3$. Let $g : z \quad z$ be a function defined by $g(x) = 3x + 2$.

Find (a) fog (b) gof

Solutions : Both fog and g of are defined as

(a) $(fog)(x) = f(g(x)) = f(3x + 2)$

$=2(3x + 2) + 3$

$=6x + 4 + 3 = 6x + 7$

(b) $(gof)(x) = g(f(x)) = g(2x + 3)$

$=3(2x + 3) + 2$

$= 6x + 9 + 2$

$= 6x + 11$

Even though fog and gof are defined and fog and gof need not be equal.

i.e. The commutative law does not hold the composition of function.

Example (2)

Let A = {1, 2, 3}, B = {x, y}

C = {a}. Let f : A   B be defined by g(1) = *x*, g(2) = *y*, *f*(3) = *x*

Let g : B   C be defined by g(x) = a, g(y) = a

Find (a) fog if possible (b) gof if possible.

Solutions :

The solution is obtained as follows

(a) (fog) (x) = *f* (g(x)) by definition But f cannot be applied on C and hence fog is meaningless.

(b) (gof) : A   C is meaningfull.

Now (gof) (x) = g(f(x), $\forall$ x   A

(gof) (1) = g(f(1)) = g(x) = a

(gof) (2) = g(f(2)) = g(y) = a

(gof) (3) = g(f(3)) = g(x) = a

Relations

4.8.1 **Relations :** A relation is a set of ordered pairs. That is, a relation is a set, each member of which is an ordered pair. If R is a relation, we write xRy and (x, y)   R.

Interchangeably and we say that x is

R-related to y if and only if xRy.

The domain of a relation R is the set of all first co-ordinates of members of R, and its range is the set of all second co-ordinates.

Formally, domain $R \quad \{x :\text{for some y}, (x, y) \quad R)$

Range $R \quad \{y :\text{for some x}, (x, y| \quad R|$

one of the simplest relations is the set of all pairs (x, y) such that *x* is a member of some fixed set A and y is a member of some fixed set B. This relation is the cartesian product of A and B and is denoted by A × B.

Thus $A{\times}B= \{(x, y) : x \quad A \text{ and } y \quad B\}$

It is evident that every relation is a subset of the cartesian product of its domain and range.

Example : Let A = {a, b, c, d} an B = {*l*, m, n}

Now every sub set of A × B is a relation from A to B.

If R = {(a, l), (b, l), (c, m)}, then the domain and range of R are {a, b, c} and {*l*, m} respectively.

4.8.2 Difference between a relation and a function.

Let A and B be two non empty sets, let

$x \quad A \text{ and } y \quad B$

A function f from A to B is defined as a subset of A × B in which an ordered pairs $(x, y) \quad A{\times}B$ occurs only one time with x as the first element.

Hence f is a subset of A × B satisfying.

The following conditions :

(a) $\forall x \quad A, (x, y) \quad f$ for some $y \quad B$

(b) if $(x, y) \quad f$ and $(x, y') \quad f$ then $y = y'$

By definition of relation, every subset of A × B is a relation, Say R from A to B.

	Function *f*		**Relation R**
(i)	The domain of *f* is equal to A	(i)	The domain of R may be a sub-set of A
(ii)	An element of A can not be associated to more than one element in B.	(ii)	An element of A may be related to more than one element in B.
(iii)	Each element of A must be related to any element in B	(iii)	Some elements of A may not be related to any element in B.

From the definitions of function and relations. We conclude that

(a) Every function is a relation

(b) Every relation is not necessarily a function.

Let us Illustrate these facts by an example.

Let A = {1, 2, 3} and B = {a, b, c, d}

Then A × B = {1, 2, 3} × {a, b, c, d}

= {(1, a), (1, b), (1, c), (1, d), (2, a), (2, b), (2, c), (2, d), (3, a), (3, b), (3, c) (3, d)}

Let us consider two sets R and r defined by

R = {(1, a), (2, b), (3, b)}

r = {(1, a), (1, b) (2, c), (2, d)}

Evidently $R \quad A\times B$ and $r \quad A\times B$

We know that every subset of A × B is a relation from A to B.

Hence R and r are relations from A to B in relation R, we find that 1 a, 2 b, 3 c. Each element of A has one and only one Image in B. Hence R is a function from A to B.

In relation r, we have $1 \quad a, 1 \quad b, 2 \quad c, 2 \quad d$.

That is, $1 \quad A$ has two Images $a, b \quad B$ also $3 \quad A$ has no Image in B that is, the domain of r, that is (1, 2) is a proper subset of A. Hence r is not a function from A to B.

Hence R is a relation as well as a function r is a relation but not a function.