Nalanda Open University

M.SC Part-1

Course : Physics

Paper : 1

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Topic- Classical Mechanics (Solution of Harmonic Oscillator using Hamiltonian Jacobi Equation)

Harmonic Oscillator

Let us apply Hamilton–Jacobi method to a harmonic oscillator. Of course, a harmonic oscillator can be easily solved using the conventional equation of motion, but this excercise would be useful to understand the basic method.

From the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, \tag{1}$$

Since we are trying to find $F(q,Q)$, we want to keep our 'favorite' variables q,Q in all equations, and remove p,P where ever we see them. We see p in H, and so we remove it by writing $p = \frac{\partial F(q,Q,t)}{\partial q}$. Then we will have the equation

$$\frac{\partial F(q,Q,t)}{\partial t} + \frac{1}{2m}\left(\frac{\partial F(q,Q,t)}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 = 0 \tag{2}$$

Now it is indeed true that we can find $F(q,Q,t)$, and thus find our required canonical transformation; all we have to do is solve the above equation. But it is a partial differential equation, and so it has a very large number of solutions! In fact, we can choose any form of the function F at $t = 0$

$$F(q,Q,0) = f(q,Q) \tag{3}$$

and then (2) will find F all all other t, thus providing a solution to the problem. When we find Q, P from F, we will pick up this time dependence; we had already seem in the last section that we expect the definitions of these variables to be time dependent.

Let us try to solve (2) . The equation involves derivatives in t and q; the variable Q is more of a spectator. First we see that we can separate the variables q, t by writing

$$F(q, Q, T) = W(q, Q) - V(Q)t \tag{4}$$

We let the t part be simple, since we can see that a first derivative in t will remove t, and there is no t elsewhere in the equation. The variable q is more complicated, since it appears in the potential term as well, so we have left an undetermined function $W(q, Q)$ in F.

Now we get

$$\frac{1}{2m}(\frac{\partial W(q,Q)}{\partial q})^2 + \frac{1}{2}m\omega^2 q^2 = V(Q) \tag{5}$$

Note that Q will be one of our canonical coordinates at the end. We are not very particular at this point about exactly which coordinate Q will be, so we might as well choose $\tilde{Q} = V(Q)$ to be our new coordinate. This will simplify the equation

$$\frac{1}{2m}(\frac{\partial \tilde{W}(q,\tilde{Q})}{\partial q})^2 + \frac{1}{2}m\omega^2 q^2 = \tilde{Q} \tag{6}$$

where $\tilde{W}(q,\tilde{Q}) = W(q,Q)$. We do not want to carry the symbol tilde along, so for convenience we rewrite variables in (6) to get

$$\frac{1}{2m}\left(\frac{\partial W(q,Q)}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 = Q \tag{7}$$

We have now used up all our choices of simplifications, and have to work to solve this equation for W. We have

$$\frac{\partial W(q,Q)}{\partial q} = \sqrt{2mQ - m^2\omega^2 q^2} \tag{8}$$

so that

$$W(q,Q) = \int dq \sqrt{2mQ - m^2\omega^2 q^2} + g(Q) \tag{9}$$

Since all we want is to get one solution to the equation, we might as well set $g(Q) = 0$ so that

$$W(q,Q) = \int dq \sqrt{2mQ - m^2\omega^2 q^2} \tag{10}$$

and

$$F(q,Q,t) = \int dq \sqrt{2mQ - m^2\omega^2 q^2} - Qt \tag{11}$$

Now we can return to our other equations

$$p = \frac{\partial F(q, Q, t)}{\partial q} \tag{12}$$

This is easily solved, since differentiating in q just cancels the integration

$$p = \sqrt{2mQ - m^2\omega^2 q^2} \tag{13}$$

The other equation is

$$P = -\frac{\partial F(q, Q, t)}{\partial Q} = -m \int \frac{dq}{\sqrt{2mQ - m^2\omega^2 q^2}} + t \tag{14}$$

Eq. (13) gives

$$Q = \frac{1}{2m}[p^2 + m^2\omega^2 q^2] \tag{15}$$

Eq.(14) gives

$$P = -\frac{1}{\omega} \sin^{-1} \sqrt{\frac{m\omega^2 q^2}{2Q}} + t \tag{16}$$

Substituting the value of Q found in (15) we get

$$P = -\frac{1}{\omega} \sin^{-1} \frac{m\omega q}{\sqrt{p^2 + m^2\omega^2 q^2}} + t \tag{17}$$

We can also rewrite this as

$$P = -\frac{1}{\omega}\tan^{-1}\frac{m\omega q}{p} + t \qquad (18)$$

Thus we see that by using the method of generating functions, we could arrive at the canonical transformation that we needed without guesswork at any stage.

But one may still ask: what was the point of obtaining the new variables Q, P. The answer is, that in doing all this we have solved the dynamical equation of the harmonic oscillator, though we have not yet explicitly realized this. Since the new Hamiltonian $K = 0$, we know that Q, P will be constant in time. From (16) we see that we have an equation that gives q in terms of t, in terms of two constants Q, P. All we have to do is the algebra needed to invert (16). Doing this algebra, we get

$$q = -\sqrt{\frac{2Q}{m\omega^2}}\sin[\omega(P-t)] \qquad (19)$$

The two constants Q, P thus give, with some scalings, the amplitude and the phase of the oscillator. Thus we have found the sinusoidal vibrations of the oscillator, complete with its two arbitrary constants of motion, without having explicitly written an equation of the form $\ddot{q} + \omega^2 q = 0$. Of course this was a simple problem, and we already knew the solution. But in a more complicated case, it may well be that the solution obtained by this Hamilton-Jacobi way might be easier than the solution obtained by directly solving the equation of motion.