ROOTS OF NON-LINEAR EQUATIONS

Lecture 1: (a) Method of tabulation.

(b) Bisection method.

1. Introduction

In this section, we solve the equation of the form $f(x) = 0$. The equation $f(x) = 0$, will be called algebraic or transcendental according as *f(x)* is purely a polynomial in x or contains some other functions, such as trigonometric, logarithmic or exponential functions etc. For example

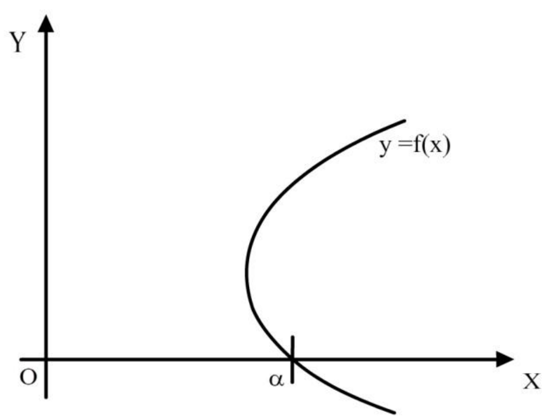
$$x^3 - x + 2 = 0$$

is an algebraic equation and

$$\sin(x) + x^2 - \log x = 0$$

is a transcendental equation. Normally, it is assumed that f (x) is a continuous function.

By finding the solution of the equation $f(x) = 0$, we mean to find those values of x for which f(x) = 0. Such values of x are called the roots or zeros of the equation $f(x) = 0$. In this section, we will only concentrate on finding the real roots of the equation $f(x) = 0$. Geometrically, by solving $f(x) = 0$, we wish to find those points where the graph of $f(x)$ crosses the x-axis. In other words, we are finding the points of intersection of the graph of $f(x)$ with the x-axis (see figure below).



2. **Location of Roots**

There are various methods of solving the equation of the form $f(x) = 0$, whether algebraic or transcendental. With the help of these methods, we first find an approximate value of the root of the equation and then successively improve it. Here, we discuss only one method, namely the method of tabulation, to obtain the location of a root.

Method of Tabulation: The method of tabulation is the application of the Bolzano's theorem on continuity which states that **if the function $\boldsymbol{f(x)}$ is continuous in the closed interval $[\boldsymbol{a}, \boldsymbol{b}]$ and if $\boldsymbol{f(a)}$ and $\boldsymbol{f(b)}$ are of opposite signs, then there exists at least one real root of** $f(x) = 0$ **between $\boldsymbol{a}$ and $\boldsymbol{b}$.**

Geometrical Interpretation of the Method of Tabulation: If the graph of a continuous function lies above the x-axis at one end of an interval [a, b] and below the x-axis at another end, then it must cross the x-axis somewhere in between at least once.

3. **Method of Bisection**

This method is also based on Bolzano's theorem on continuity. Let f(x) = 0 has a root in [a, b], the function f(x) being continuous in [a, b]. Then, f (a) and $f(b)$ are of opposite signs, i.e., $f(a).f(b) < 0$.

Let $x_1 = \frac{a+b}{2}$, the middle point of $[a, b]$. If $f(x_1) = 0$, then $x_1$is the root of $f(x) = 0$. Otherwise, either $f(a).f(x_1) < 0$, implying that the root lies in the interval $[a, x_1]$ or $f(x_1).f(b) < 0$, implying that the root lies in the interval $[x_1, b]$. Thus, the interval is reduced from $[a, b]$ to either $[a, x_1]$ or $[x_1, b]$. We rename it $[a_1, b_1]$.

Let $x_2 = \frac{a_1+b_1}{2}$, the middle point of $[a_1, b_1]$. If $f(x_2)= 0$, then x_2 is the root of $f(x) = 0$. Otherwise, either $f(a_1).f(x_2) < 0$ implying that the root $\in [a_1, x_2]$ or $f(x_2).f(b_1) < 0 \Rightarrow$ the root $\in [x_2, b_1]$ and so on. We continue in this manner and the process is repeated until the root is obtained to the desired accuracy.

Example 1: Find by bisection method, a real positive root of $2x - \log_{10}(x) = 7$.

Solution. We first get an approximate location by the method of tabulation.

Here $f(x) = 2x - \log_{10}(x) - 7$. Now $f(1) = -5,$ $f(2) = -3.3,$ $f(3) = -1.477,$ $f(4) = 0.3979 \Rightarrow f(3).f(4) < 0$, so that a root α lies in (3,4).

Computation of α $(3 < \alpha < 4)$

n	a_n(-ve)	b_n(+ve)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	3	4	3.5	- 0.5441
1	3.5	4	3.75	- 0.0740
2	3.75	4	3.875	0.1617
3	3.75	3.875	3.8125	0.0438
4	3.75	3.8125	3.7813	- 0.0151
5	3.7813	3.8125	3.7969	0.0143
6	3.7813	3.7969	3.7891	- 0.0004
7	3.7891	3.7969	3.7930	0.0070
8	3.7891	3.7930	3.7910	0.0033
Check 9	3.7891	3.7910	3.7910	

In the 8th step, a_n, b_n and x_{n+1} are equal to 3- significant figures. Therefore $\alpha =$ **3.79.** correct to three significant figures.

Example 2: Find by bisection method, a real positive root of $e^{-x} = \sin(x)$.

Solution. We first get an approximate location by the method of tabulation.

Here $f(x) = e^{-x} - \sin(x)$. Now $f(0) = 1,\ f(1) = -0.47,\ f(0.5) = 0.13, f(0.6) = -0.016$ $\Rightarrow f(0.5).f(0.6) < 0$, so that a root α lies in (0.5,0.6).

Computation of α $(0.5 < \alpha < 0.6)$

n	a_n(-ve)	b_n(+ve)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	0.5	0.6	0.55	0.05426
1	0.55	0.6	0.575	0.01897
2	0.575	0.6	0.5875	0.001433
3	0.5875	0.6	0.593750	-0.007221
4	0.5875	0.593750	0.590625	-0.002899
5	0.5875	0.590625	0.589063	-0.000735
6	0.5875	0.589063	0.588281	0.000349
7	0.588281	0.589063	0.588672	-0.000193
8	0.588281	0.588672	0.588477	0.000078
9	0.588477	0.588672	0.588574	-0.000058
10	0.588517	0.588574	0.588526	0.000010
Check 11	0.588526	0.588574	0.588550	

In the 10th step, a_n, b_n and x_{n+1} are equal to 3- significant figures. Therefore $\alpha = 0.589$ correct to three significant figures.

Exercise 1: Find a real root of $x^3 + x^2 + x + 7 = 0$ by the method of bisection, correct to three significant figures. (Ans. –2.11)

Exercise 2: Find a real root of $\sin(x) = 10(x - 1)$ by the method of bisection, correct to three significant figures. (Ans. 1.09)