ROOTS OF NON-LINEAR EQUATIONS

Lecture 4: Newton Raphson method.

7. Newton Raphson Method

When the derivative of $f(x)$ is of the simple form, the real root (non-repeated) of the equation $f(x) = 0$, can be computed rapidly by a process known as the **Newton Raphson method.** Usually the problem is to find a recurrence relation which enables us to find out a sequence $\{x_n\}$ converging to the desired root α.

Let x_0 be an approximation of the root of $f(x) = 0$, whose real root is α. Thus, $\alpha = x_0 + h$, where h is the correction (small) to be applied to x_0 to give the exact value of the root. Therefore,

$$f(\alpha) = f(x_o + h) = 0$$

By Taylor series expansion we get,

$$f(x_o) + hf'(x_o) + \frac{h^2}{2!} f''(x_o) + \cdots = 0$$

Since h is small, (neglecting higher orders of $h, i.e., h^2, h^3, etc.$) we get,

$$f(x_0) + hf'(x_0) \approx 0 \quad \Rightarrow h = -\frac{f(x_0)}{f'(x_o)}$$

Substituting this value of h in $\alpha = x_0 + h,$ we get a better approximation to the root a of $f(x) = 0$ as

$$x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$$

Therefore, the successive approximations are

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

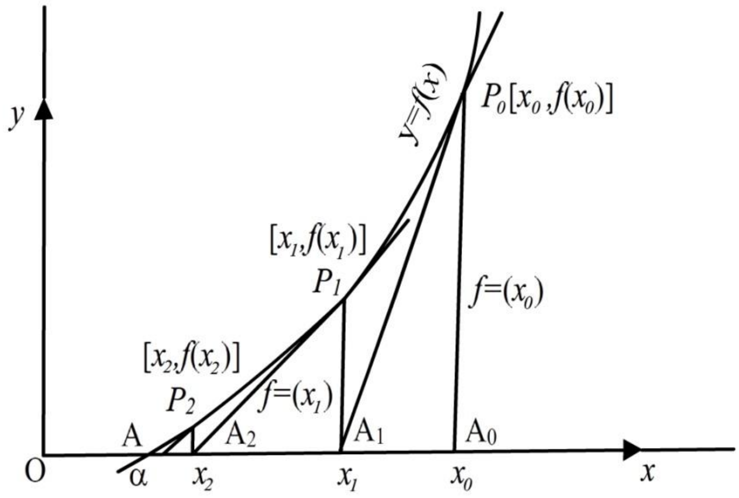
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

……………….

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This formula is known as the iteration formula for Newton Raphson Method.

Newton Raphson method: Geometrical significance



The figure represents a magnified view of the graph $y = f(x)$ where it crosses the X-axis at $x = \alpha$. Let x_o be the initial approximation at the starting point. To calculate the first approximation we replace the graph at $P_0(x_{0,}y_o = f(x_o))$ by the tangent at $P_{o.}$ This tangent will intersect the x-axis at some point A_1 whose abscissa is $OA_1 \ = x_1$.

Then, from $\Delta P_o A_1 A_o = \frac{PoAo}{A_1 A_o} = \frac{f(x_0)}{x_0 - x_1} = \frac{yo}{x_0 - x_1}$

But tan $P_0 A_1 A_o$= Slope of the tangent at P_o=$f'(x_o)$

Therefore,

$$f'(x_0) = \frac{f(x_o)}{x_o - x_1} \Rightarrow x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$$

Using x_1 as starting point, the tangent at $P_1[x_{1,}f(x_1)]$ will give x_2=$x_1 - \frac{f(x_1)}{f'(x_1)}$

Hence OA_1, OA_2 , … are successive approximations to the desired root.

Note 1: The method fails if f '(x) = 0 or very small in the neighbourhood of the root.

Note 2: The sufficient condition for convergence of Newton-Raphson method is $|f(x) f''(x)| < [f'(x)]^2$

Note 3: The Newton Raphson method is said to have a quadratic rate of convergence.

Example 8. Find by Newton-Raphson Method the real root of

$$3x - \cos x - 1 = 0.$$

Solution: Let $f(x) = 3x - \cos x - 1$. Since $f(0) = -2, f(0.5) = -0.37, f(0.7) = 0.34,$

one real root of f(x) = 0 lies between 0.5 and 0.7.

Now $f'(x) = 3 + \sin x$ and $f'(0.5) = 3.48.$

Taking $x_0 = 0.5$, the successive approximations of the root are computed in the following table:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	0.5	-0.37	3.48	0.1063	0.6063
1	0.6063	-0.00286	3.56983	0.000801	0.607101
2	0.607101	-0.00000231	3.570489	0.00000064	0.60710164
3	0.60710106	-0.00000017	3.570489	0.00000003	0.60710163

Therefore, 6.0710 is the root of f(x) = 0, correct up to five decimal places.

Example 9: Find a real root of $x^x + x - 4 = 0,$ by Newton-Raphson method, correct to six decimal places.

Solution: Let $f(x) = x^x + x - 4$ and $f'(x) = x^x(1 + \ln x) + 1$

Now, $f(1) = -2,\ f(1.5) = -0.66,\ f(1.6) = -0.27,\ f(1.7) = 0.16$.

Therefore, f(x) = 0 has a root between 1.6 and 1.7. Also, f '(1.6) = 4.12.

Taking x_0 = 1.6, the successive iterations are computed in the following table:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x)}{f'(x)}$	$x_{n+1} = x_n + h_n$
0	1.6	-0.27	4.12	0.066	1.666
1	1.666	0.0065	4.5352	-0.00143	1.66457
2	1.66457	0.0000318	4.525536	-0.000007	1.664563
3	1.664563	0.0000002	4.5254887	-0.000000004	1.664563

Thus, 1.664563 is a root of $f(x) = 0,$ correct up to six decimal places.

Example 10: Find the cube of 10, that is, $\sqrt[3]{10}$, correct to 4- significant figures.

Solution. Let $x=\sqrt[3]{10}$, then $x^3-10=0$. Let $f(x)=x^3-10 \Rightarrow f'(x)=3x^2$. Hence Newton Raphson's iterative formula gives

$$x_{n+1}=x_n-\frac{x_n^3-10}{3x_n^2}=\frac{2x_n^3+10}{3x_n^2}\Rightarrow x_{n+1}=\frac{1}{3}\left(2x_n+\frac{10}{x_n^2}\right)$$

Since $8^{1/3}=2$ and $27^{1/3}=3$, we may take initially $x_n = 2.2$. Therefore,

$$x_1=\frac{1}{3}\left(2x_0+\frac{10}{x_0^2}\right)=\frac{1}{3}\left(2\times 2.2+\frac{10}{(2.2)^2}\right)=2.15537$$

$$x_2=\frac{1}{3}\left(2\times 2.15537+\frac{10}{(2.15537)^2}\right)=2.15444$$

$$x_3=\frac{1}{3}\left(2\times 2.15444+\frac{10}{(2.15444)^2}\right)=2.15443$$

$$x_4=\frac{1}{3}\left(2\times 2.15443+\frac{10}{(2.15443)^2}\right)=2.15443$$

Therefore, $\sqrt[3]{10}=2.154$, correct to 4-significant figures.

Example 11: Find a real root of $\log_e(x) = \cos(x)$ by Newton Raphson method correct to 6-significant figures.

Solution. Here $f(x) = \log_e(x) - \cos x$ and $f'(x) = \frac{1}{x} + \sin x$. Now $f(1) = -0.54$, $f(2) = 1.1$, therefore one root of $f(x) = 0$ lies between 1 and 2. Let x_0 = 1.5 be the initial approximation. Then $f'(1.5) = 1.6$. We then calculate the successive approximations as follows:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	x_{n+1}
0	1.5	0.33472791	1.6641617	- 0.2011391	1.2988609
1	1.2988609	- 0.0071085944	1.7331582	0.0041014	1.3029624
2	1.30229624	$- 2.773.350 \times 10^{-6}$	1.7318283	0.0000021	1.3029640
3	1.3029640	$- 2.1 \times 10^{-9}$	1.7318277	0.0000000	1.3029640

Thus, 1.30296 is a root of the given equation correct to 6- significant figures.

Exercise 7: Find a real root of $\sinh(x) - x = 0$ by Newton Raphson method, correct to three significant figures. (Ans. 0.100)

Exercise 8: Find a real root of $\tan(x) - \tanh(x) = 0$ by Newton Raphson method, correct to three significant figures. (Ans. 0.619)