Nalanda Open University

M.sc Part-I

Course : Mathematics

Paper- V

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UNIT I

Contents : Boolean Algebra and its properties, Lattice, Complemented Lattice, Distributive Lattice, Complete Boolean Algebra, Sub-Algebra, Boolean Homomorphism, Boolean Ideal, Min Term, Max Term, Sum of product form, product of sum form, complete sum of product form, solved examples

1. Boolean Algebra and some general properties

1.1 Introduction -:

The concept of Boolean algebra was introduced by George Boole, An English logician and mathematician. It is being used tremendously in computer designers, Computer analysis, Electric networks and others for last so many years.

1.2 Definitions:

In this section we give some of the useful definitions.

Boolean Algebra:- By a Boolean algebra we understand a set B with at least two distinct elements 0 (zero) and 1(unit), endowed with binary operations $\wedge$ (called meet) and $\vee$ (called join) and a unary operation $'$ (called complement) satisfying the following conditions for every x, y, z $\in$ B.

Also we read $x \wedge y$ as meet of x and y and we read $x \vee y$ as join of x and y.

We read x' as complement of x.

(A_1) $0' = 1$, $1' = 0$

(A_2) $x \wedge 0 = 0$, $x \vee 1 = 1$

(A_3) $x \wedge 1 = x$, $x \vee 0 = x$

(A_4) $x \wedge x' = 0$, $x \vee x' = 1$

(A_5) $(x')' = x$ Involution Law

(A_6) $x \wedge x = x$, $x \vee x = x$

(A_7) $(x \wedge y)' = x' \vee y'$, $(x \vee y)' = x' \wedge y'$

(A_8) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$

(A_9) (i) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

(A_9) (ii) $x \vee (y \vee z) = (x \vee y) \vee z$

(A_{10}) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(A_{11}) $x \wedge (x \vee y) = x$

$x \vee (x \wedge y) = x$

There operations $\wedge$, $\vee$ and $'$ are also called Boolean operations

(A_7) is De Morgan laws, (A_8) is commutative laws.

(A_9) is called Associative Law, (A_{10}) is called Distributive Law, (A_{11}) is called absorption Laws.

Principal of duality:- If the pair 0 and 1 are inter changed and similarly the meet ($\wedge$) and join ($\vee$) are interchanged then conditions from (A_1) to (A_{11}) remain the same as they are permuted among themselves and this is called the principle of duality.

Clopen Set:- Let A be a sub-set of a topological space x such that A is simultaneously open and closed then A is called clopen set.

Regular Open set:- Let X be a topological space then an open set G in X is said to be a regular open if $G = (\bar{G})^0$ That is G equals the interior of the closure of G itself.

Difference between any two elements:- Let x, y be element of a Boolean Algebra B then difference between x and y is denoted and defined as below:-

$x - y = x \wedge y'$

Boolean Sum of x and y :- Let x, y be any two element s of a Boolean algebra B then the Boolean sum of x and y is denoted by $x \Delta y$ and we defined it as $x \Delta y = (x - y) \vee (y - x)$

$x \Rightarrow y$ and $x \Leftrightarrow y$:- For elements x and y of a Boolean algebra B we defined $x \Rightarrow y$ as $x' \vee y$, and $x \Leftrightarrow y$ as $(x \Rightarrow y) \wedge (y \Rightarrow x)$.

x / y :- For elements x and y of a Boolean algebra B we defined $x / y = x' \wedge y'$. The operation / is called the siroke (sheffer). In logical context this operation is also known as binary rejection (neither x nor y)

Expression of complement, meet and join in terms of the stroke:-

$x' = x / x$

$x \wedge y = (x / x) / (y / y)$

$x \vee y = (x / y) / (x / y)$

1.3 Theorems

Theorem (1.3) i :- Let P(X) be the set of all subsets of the set X. Then P(X) is a Boolean algebra with $0 = \varphi$, $1 = X$ and with the Boolean operations $\wedge$ (meet), $\vee$ (join) and $'$ defined by $A \wedge B = A \cap B$, $A \vee B = A \cup B$, $A' = X - A$ $\forall$ A, B $\in$ P(X) i.e, A, B $\subseteq$ X.

Observation:- Since from set theoratic operations we know that $\varphi' = X$, $X' = \varphi$, $A \cap \varphi = \varphi$,

$A \cup X = X$, $A \cap X = A$, $A \cup \varphi = A$, $A \cap A' = \varphi$, $A \cup A' = X$, $(A')' = A$ $\forall$ A.

$A \cap A = A$, $A \cup A = A$

$(A \cap B)' = A' \cup B'$, $(A \cup B)' = A' \cap B'$

$A \cap B = B \cap A$, $A \cup B = B \cup A$

$A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (A \cup B) = A$, $A \cup (A \cap B) = A$, $\forall$ A, B, C $\in$ P(X)

Therefore P(X) is a Boolean algebra.

Theorem (1.3) ii :- Prove that a Clopen set having o = φ and 1 = X is a Boolean algebra or prove that a close or open set G having o = φ and 1 = X is a Boolean algebra.

Proof :- Let X be a topological space. Again let G be a subset of X which is simultaneously open and close with o = φ, 1 = X.

Again let G has Boolean operations $\wedge$ (meet), $\vee$ (join) and ' defined by $A \wedge B = A \cap B$, $A \vee B = A \cup B$, $A' = X - A$.

Then as we know from the set theoratic operation that the above Boolean operations satisfies all the axioms for G to be a Boolean algebra. Thus G will become a Boolean algebra.

Theorem (1.3) iii :- (1) If $x \vee y = x$ for all $x \in A$, then y = 0

(2) $x \wedge y = x$ for all $x \in A$, then y = 1.

Observation:- (1) Take $x = 0 \in A$ then $0 \vee y = 0$. But $0 \vee y = y \Rightarrow y = 0$

(2) Let us take $x = 1 \in A$ then $1 \wedge y = 1$, but $1 \wedge y = y$, hence y = 1

Note : It should be kept in our mind that the assertion (2) is the dual of the (1).

Theorem (1.3) iv :- If x, y $\in$ A are such that $x \wedge y = 0$ and $x \vee y = 1$ then $y = x'$

Proof :- $y = 1 \wedge y = (x \vee x') \wedge y = (x \wedge y) \vee (x' \wedge y) = 0 \vee (x' \wedge y)$

$= (x' \wedge x) \vee (x' \wedge y) = x' \wedge (x \vee y) = x' \wedge 1 = x'$

Theorem (1.3) v :- State and prove laws of Absorption

Statement :- It states that for all x, y in A,

(i) $x \vee (x \wedge y) = x$
(ii) $x \wedge (x \vee y) = x$

Proof :- $x \vee (x \wedge y) = (x \wedge 1) \vee (x \wedge y) = x \wedge (1 \vee y) = x \wedge 1 = x$

Theorem (1.3) vi :- Prove that for x, y, z $\in$ A then

$$x \wedge (y - z) = (x \wedge y) - (x \wedge z)$$

Proof :- $(x \wedge y) - (x \wedge z) = (x \wedge y) \wedge (x \wedge z)' = (x \wedge y) \wedge (x' \vee z')$

$= (x \wedge y \wedge x') \vee (x \wedge y \wedge z') = (x \wedge x' \wedge y) \vee (x \wedge y \wedge z')$

$= (0 \wedge y) \vee (x \wedge y \wedge z') = 0 \vee (x \wedge y \wedge z') = x \wedge y \wedge z'$

$= x \wedge (y \wedge z') = x \wedge (y - z)$

2. Order relation in Boolean Algebra, Complemented Distributive Lattice, Complete Boolean Algebra.

2.1 Introduction :- Here order relation is in fact partial order relation.

2.2 Definition :-

Lattice : A partially ordered set (L, $\leq$) is called a lattice if each pair set (a, b) of elements of L has a least upper bound and a greatest lower bound in L.

Complemented Lattice : It is a bounded lattice (with least element 0 and greatest element 1), in which every element a has a complement i.e, an element b such that $a \vee b = 1$ and $a \wedge b = 0$. b is also denoted by a^1 and is called the complement of a.

Thus we can conclude that a lattice is complemented iff every element of it has a complement.

Distributive Lattice : A lattice L is said to be distributive if it satisfies the following conditions:-

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Complete Boolean Algebra : A Boolean algebra with the property that every subset of it has both a supremum and an infimum is called a complete Boolean algebra.

2.3 Theorems:-

Theorem (2.3) i :- A Boolean algebra is a complemented distributive lattice.

Proof :- Let B be a Boolean algebra with the Boolean operations $\wedge$ (meet), $\vee$ (join) and $'$ and whose zero and unit are denoted by 0 and 1.

We now show that a partial order relation $\leq$ can be defined in B in such a way that (B, $\leq$) becomes a lattice in which for all x, y $\in$ B, $x \wedge y$ and $x \vee y$ are the greatest – lower bound and the least – upper bound of (x, y).

Since by law of absorption $x \wedge y = x$ and $x \vee y = y$ are equivalent.

Or if $x \wedge y = x$ then $x \vee y = (x \wedge y) \vee y = (y \wedge x) \vee y = y \vee (y \wedge x) = y$

And if $x \vee y = y$ then $x \wedge y = x \wedge (x \vee y) = x$ (y the axiom of Boolean at algebra)

We now define $x \leq y$ to mean that either $x \wedge y = x$ or $x \vee y = y$.

Then $x \wedge x = x$ then we have $x \leq x$ for every x $\in$ B.

Thus $\leq$ is reflexive in B .

If $x \leq y$ and $y \leq x$, then $x \wedge y = x$ and $y \wedge x = y$ then

$x = x \wedge y = y \wedge x = y$ => $\leq$ is antisymmetric in B.

Again, if $x \leq y$ and $y \leq z$, so that $x \wedge y = x$ and $y \wedge z = y$, then

$x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$ so $x \leq z$

Thus $\leq$ is transitive in B.

Thus $\leq$ is a partial order relation in B and hence (B, $\leq$) is a partially ordered set.

We now show that $x \wedge y$ is the greatest – lower bound of x and y.

For this, since $(x \wedge y) \vee x = x \vee (x \wedge y) = x$ and

$(x \wedge y) \vee y = y \vee (x \wedge y) = y \vee (y \wedge x) = y$

We see that $x \wedge y \leq x$ and $x \wedge y \leq y$

Thus $x \wedge y$ is a lower bound of (x, y)

If $z \leq x$ and $z \leq y$, so that $z \wedge x = z$ and $z \wedge y = z$, then

$z \wedge (x \wedge y) =(z \wedge x) \wedge y = z \wedge y = z$, so $z \leq x \wedge y$.

Hence $x \wedge y$ is the greatest – lower bound of { x, y } in B.

Similarly, it can be proved that $x \vee y$ is the least – upper bound of { x, y } in B.

Thus the partially ordered set (B, $\leq$) becomes a lattice.

Further a lattice is said to be distributive if it has the following conditions :

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Since B is a Boolean algebra so the above two conditions are automatically satisfied.

Thus $(B, \leq)$ is a distributive lattice .

Finally, since B is a Boolean algebra so it contains distinct elements 0 and 1 and $0 \wedge x = x \wedge 0 = 0, x \vee 1$.

We have $0 \leq x \leq 1$ for every $x \in B$.

Also $x \wedge x' = 0$ and $x \vee x' = 1$ for each $x \in B$.

Thus $(B, \leq)$ is complemented distributive lattice.

Thus a Boolean algebra can be regarded as a complemented distributive lattice.

Theorem (2.3) ii :- A complemented distributive lattice is a Boolean algebra.

Proof :- Let $(B, \leq)$ be a complemented distributive lattice.

Let $x \wedge y$ stands for greatest – lower bound and $x \vee y$ denote the least – upper bound of $\{x, y\}$ for $x, y \in B$.

There exist distinct element 0 and 1 in B such that $0 \leq x \leq 1$ for every $x \in B$ and each element x of B has complement x' with the property that

$x \wedge x' = 0$ and $x \vee x' = 1$

Also since $(B, \leq)$ is a distributive lattice and $0 \leq x \leq 1$, we have $0 \wedge 1 = 0$ and $0 \vee 1 = 1$, so we have $0' = 1$ and $1' = 0$

Again since $0 \leq x \leq 1$ for every x, we have,

$x \wedge 0 = 0, x \vee 1 = 1, x \wedge 1 = x, x \vee 0 = 0$

Since B is complemented distributive lattice so by theorem x has only one complement x'.

Hence $x \wedge x' = 0$ and $x \vee x' = 1$, x is the complement of x' and $(x')' = x$.

Further we note that $x \leq y \Leftrightarrow x' \leq y'$.

Since, if $x \leq y$, then $x \wedge y' \leq y \wedge y' = 0$

Thus $y' = y' \wedge 1 = y' \wedge (x \vee x') = (y' \wedge x) \vee (y' \wedge x') = 0 \vee (y' \wedge x') = y' \wedge x'$ so $y' \leq x'$

That is $x \leq y \Rightarrow y' \leq x'$.

Now by taking complementation we also have $y' \le x' \Rightarrow x \le y$

We now prove that $(x \wedge y)' = x' \vee y'$

For this, we see that if $x' \le x$ and $y' \le z$, so that $z' \le x$ and $z' \le y$.

Then $z' \le x \wedge y$ or $(x \wedge y)' \le z$

$\Rightarrow (x \wedge y)' \le$ any upper bound of (x', y'), so $(x \wedge y)' \le x' \vee y'$

Again, since $x' \le (x \wedge y)'$ and $y' \le (x \wedge y)'$ which are equivalent to $x \wedge y \le x$ and $x \wedge y \le y$ are clearly true so $x' \vee y' \le (x \wedge y)'$

Thus $(x \wedge y)' = x' \vee y'$

Similarly it can be established that $(x \vee y)' = x' \wedge y'$

Thus we find that all the conditions to be Boolean algebra are satisfied.

Thus the complemented distributive lattice is a Boolean algebra.

Corollary (2017) (2.3) iii :- In a complemented distributive lattice B, each element has only one complement or, In a Boolean algebra the complement of an element is unique.

Observation:- For x, y in B let $x \wedge y$ and $x \vee y$ are respectively greatest lower bound (g.l.b) and least upper bound (l.u.b) of $\{x, y\}$

If possible let for a moment x' and x^* are two complements of the same element x which is in B.

Then $x \wedge x' = 0$ and $x \vee x' = 1$ (by the axiom)

Also $x \wedge x^* = 0$ and $x \vee x^* = 1$

Then $x^* = x^* \wedge 1 = x^* \wedge (x \vee x') = (x^* \wedge x) \vee (x^* \wedge x')$

$= 0 \vee (x^* \wedge x') = x^* \wedge x'$

Hence $x^* \le x'$

Also reversing the roles of x' and x^* we can get obviously $x' \le x^*$

Hence $x' = x^*$. Thus x' and x^* are not different.

Corollary (2017) (2.3) iv :- Relation precede $(x \le y)$ in a Boolean algebra B is a partial order relation.

Observation:- Let B be a Boolean algebra with the Boolean operations $\wedge$, $\vee$ and $'$. Also let zero and unit of B are denoted by 0 and 1.

We now defined $x \le y$, for x, y in B, to mean that either $x \wedge y = x$ or $x \vee y = y$

Then we find that $x \wedge x = x$ (see the axiom)

So we have $x \le x$ for every x in B.

Thus $\le$ is reflexive in B.

Also, if $x \le y$ and $y \le x$ then $x \wedge y = x$ and $y \wedge x = y$ then

$x = x \wedge y = y \wedge x = y$ (see the axiom)

Thus $\le$ is anti symmetric.

Finally, if $x \le y$ and $y \le z$

So that $x \wedge y = x$ and $y \wedge z = y$ then

$x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$

Hence $x \le z$

Thus $\le$ is transitive also in B

Thus $\le$ is partial order relation (P.O.R) in B.

3. Sub algebras of Boolean Algebra

3.1. Introduction :- It relates the study of the subset o0f Boolean algebra.

3.2 Definition:-

Boolean sub algebra:- A non empty subset B of a Boolean algebra A is a Boolean sub algebra if it contains 0 and 1 and it closed under the Boolean operation given as :

(i) $0 \in B$, $1 \in B$

(ii) If $x, y \in B$, then $x \wedge y \in B$, $x \vee y \in B$, $x' \in B$.

3.3 Theorems:-

Corollary (3.3) i :- Every Boolean sub algebra B of a Boolean algebra A contains 1.

Proof :- Since B is non empty, so let $x \in B$. then $x' \in B$. and so $x \vee x' \in B$.

But $x \vee x' = 1$. Hence $1 \in B$.

Thus we can deduce the existence of $1 \in B$ from the non emptiness of B.

Theorem (2016) (3.3) ii :- The intersection of two sub algebras of a Boolean algebra B is a Boolean sub algebra.

Proof :- Let A_1 and A_2 be any two sub algebras of a Boolean algebra A.

Let 0 be the zero of A and 1 be the unit of A.

Thus by the definition of sub algebras $0 \in A_1$, $0 \in A_2$.

Similarly $1 \in A_1$, $1 \in A_2$

Thus . $0 \in A_1 \cap A_2$ and $1 \in A_1 \cap A_2$

Let $x, y \in A_1 \cap A_2 \Rightarrow x, y \in A_1$ and $x, y \in A_2$

But A_1 and A_2 are sub algebras.

Thus $x \wedge y$, $x \vee y$ and x' belong to each of A_1 and A_2

Thus $x \wedge y$, $x \vee y$ and x' belong to $A_1 \cap A_2$

Therefore $A_1 \cap A_2$ is a sub algebra of A.

4. Boolean Homomorphism, Boolean Ideal

4.1. Introduction: - This relates the study of a structure preserving mapping between Boolean algebras.

4.2 Definition:-

Boolean Homomorphism:- Let A and B be any two Boolean algebras then a mapping $f : A \rightarrow B$ from A to B is called A Boolean Homomorphism if for all $x, y \in A$ we have:

(i) $f(x \wedge y) = f(x) \wedge f(y)$
(ii) $f(x \vee y) = f(x) \vee f(y)$
(iii) $f(x^1) = (f(x))'$

Monomorphism:- If $f : A \rightarrow B$ is one – one into.

Epimorphism:- If $f : A \rightarrow B$ is onto.

Isomorphism:- If $f : A \rightarrow B$ is one – one, onto.

Automorphism:- If $f : A \rightarrow B$ is one – one onto.

Isomorphic:- If $f : A \to B$ is an isomorphism then A and B are called Isomorphic.

Kernel of Boolean Homomorphism:- Let f be a Boolean Homomorphism from Boolean algebra A to a Boolean algebra B. We defined the kernel of f, denoted by ker f and ker f = $\{ x \in A : f(x) = 0 \}$

Boolean Ideal (In a Boolean algebra):- A subset M of a Boolean algebra is called Boolean ideal if the following two conditions are satisfied

(i) $0 \in M$
(ii) $x, y \in M \Rightarrow x \vee y \in M$
(iii) $x \in M$ and $y \in A \Rightarrow x \wedge y \in M$

4.3 Theorems:-

Theorem (4.3) i :- If $f : A \to B$ is a Boolean Homomorphism then

$f(0) = 0$ and $f(1) = 1$

Proof :- Since for any element x in A we have $x \wedge x' = 0$.

Thus $f(0) = f(x \wedge x') = f(x) \wedge f(x') = f(x) \wedge (f(x))' = 0$

Also since $x \vee x' = 1 = f(x) \vee f(x') = f(x) \vee (f(x))' = 1$

Thus from above we can conclude that 0 and 1 are contained in the range of every Boolean Homomorphism.

Theorem (4.3) ii :- The Kernel of every Boolean Homomorphism is a proper Boolean ideal.

Proof :- Let A and B be any two Boolean algebras and $f : A \to B$ from A to B be a Boolean Homomorphism.

Let M = ker f = $\{ x \in A : f(x) = 0 \}$ as $f(0) = 0$

Also, if $x, y \in M \Rightarrow f(x) = 0, f(y) = 0$

But $x, y \in M$ then $x, y \in A$ and A is a Boolean algebra and $f : A \to B$ is Homomorphism

Thus $x \vee y \in A$ and $f(x \vee y) = f(x) \vee f(y) = 0 \vee 0 = 0$

Thus $x \vee y \in M$

Again if $x \in M$ and $y \in A$ then $f(x) = 0$ and

$x \wedge y \in A$ and $f(x \wedge y) = f(x) \wedge f(y) = 0 \wedge f(y) = 0 \Rightarrow x \wedge y \in M$

hence M is a Boolean ideal

Since $1 \in A$ but $f(1) \neq 0 \Rightarrow 1 \notin M$.

Therefore M is a proper ideal.

Theorem (4.3) iii :- Let f be a Boolean Homomorphism then f is one – one iff ker f = {0}

Proof :- Let f is one – one and f(x) = 0 then f(x) = f(0)

Thus $x = 0 \Rightarrow \ker f = \{0\}$

Conversly :- Let ker f = {0} and if

$f(x) = f(y)$ then $f(x+y) = f(x) + f(y) = f(x) + f(x) = 0 + 0 = 0$

Also if $x + y \in \ker f = \{0\} \Rightarrow x + y = 0 \Rightarrow x = - y = y \Rightarrow x = y$

That is $f(x) = f(y) \Rightarrow x = y \Rightarrow f$ is one – one.

5. Min Term, Max Term, Sum of product form, product of sum form, complete sum of product form .

5.1. Introduction:-

It is a function in excel multiplies range of cells or a arrays and returns the sum of products.

5.2. Definition:-

Min term:- It is the term that is true for a minimum number of combination of inputs. That is true for only one combination of inputs . since AND gate also gives true only when all of its inputs are true so we can say that min terms are AND of input combinations. In like word min term is nothing but each indivisual product term in standard sum of product.

Sum of product form:- It is a form of an expression in Boolean algebra in which different product terms of input are being summed together. This product is not arithmetical multiply but it is Boolean logical AND and the sum is Boolean logical OR in short form we write it sop.

complete sum of product :- A Boolean expression $E = E(x_1, x_2, ...,x_n)$.is said to be complete sum of products expression if E is a sum of products expression where each product p (say) involves all the n variables is called a minterm, and there is a maximum of 2^n such products for n variables. Also every non zero Boolean expression $E = E(x_1, x_2, ...,x_n)$.is equivalent to a complete sum of products expression and such a representation is unique and this unique representation of E is called the complete sum of product form of E.

product of sum:- It is the logical expression Boolean algebra, where all the input terms are OR ed first and the results are PRODUCT ed,

The summing operation is not the arithmetic or sum but it is logical OR operation. As this reverse operation for what SOP (sum of products).

It is even termed as duel SOP. The name for product of sum is the conjuctive normal form where logical AND is performed. In short we write it POS.

Max Term:- The very word max term stands for a maximum number of combinations of input terms or even defined as false for only one combination. So the term SOP (sum of product) is also defined as the expression which is complete of max terms.

5.3. Results (or Theorems):- Here in this section we give working rules to transform a Boolean expression E into the sum of products and then to complete sum of products.

W. R 1:- The input is a Boolean expression E. The output is a sum of products expression equivalent to E. For this following steps are taken:

Step i:- Use De – Morgan's laws and involutions to move the complement operation into any parenthesis until finally the complement operation only applies to variables. Then E will consists only of sums and products of literals.

Step ii:- We use distributive operation to next transform E into a sum of products.

Step iii:- We use commutative, idempotent and complement laws to transform each product in E into 0 (zero) or a fundamental product.

Step iv:- We use absorption and identity laws to finally transform E into a sum of products expression.

W. R 2:- The input is a Boolean sum of product expression $E = E(x_1, x_2, ..., x_n)$. The output is a sum of product expression equivalent to E.

Step i:- Firstly find a product p in E which does involve the variable x_i and then multiply P by $x_i + x'_i$. Delete any repeated products (This is possible. Since $x_i + x^1_i = 1$, and p + p = p)

Step ii:- We repeat the step i until every product p in E is a min term, that is every product p involves all the variables.

6. Solved examples:-

Example 1(2016):- The collection of all Lebesgue measurable sets of reals is a Boolean algebra.

Solution : Since the intersection of any two Lebesgue measurable sets is Lebesgue measurable.

Similarly the union of any two Lebesgue measurable sets is Lebesgue measurable.

Also the complement of a L- measurable set is L- measurable. Also φ is a real line R are Lebesgue measurable.

Thus the collection of all Lebesgue measurable sets of reals is a Lebesgue measurable.

Example 2 (2016):- Let X be a topological space then E the collection of all open sets in X is Boolean operations $\wedge$, $\vee$ and $'$ defined by $A \wedge B = A \cap B$, $A \vee B = A \cup B$, $A' = X - A$, $\forall$ A, B $\in$ E.

Example 3:- For any Boolean algebra (A, $\leq$), the subset consisting of the top and bottom elements alone is always a sub-algebra. Since the operation $\wedge$, $\vee$ and $'$, applied to top and bottom element, then again we get top or bottom element.

Also every Boolean algebra is a sub-algebra itself.

Example 4:- Let A be a non empty subset of a set X then power set P(A) of A and P(X) of X are Boolean algebras.

Also $P(A) \subseteq P(X)$.

Thus P(A) is not a Boolean sub-algebra of P(X).

Example 5:- Find the max term of Boolean algebra P(A) where A = { x, y }.

Solution : Possible max term are $x + y$, $x + y'$, $x' + y$ and $x' + y'$.

Example 6 (2018):- Find the max term of Boolean algebra P(A) where A = { a, b, c }.

Solution : Since P(A) contains 2^3 elements = 2 . 2 . 2 = 8 elements.

We give below a table which clearly describes the max term of P(A). Let L = lower term and H = higher term.

a b c	max term
L L L	$M_0 = a + b + c$
L L H	$M_1 = a + b + c'$
L H L	$M_2 = a + b' + c$
L H H	$M_3 = a + b' + c'$
H L L	$A_4 = a' + b + c$
H L H	$A_5 = a' + b + c'$
H H L	$A_6 = a' + b' + c$
H H H	$A_7 = a' + b' + c'$

Example 7:- Express each Boolean expression E (x, y, z) as a SOP (sum of product) and its complete sum of product.

(a) $E = x (x y' + x' y + y' z)$
(b) $E = z (x' + y) + y'$. (Nov. 2018)

Solution : We firstly use working rule 1 to express E as a sum of product, and then use working rule 2 to express E as a complete sum of product.

(a) First we have $E = x x y' + x x' y + x y' z = x y' + x y' z$

Then $E = x y' (z + z') + x y' z = x y' z + x y' z' + x y' z = x y' z + x y' z'$

(b) First we have $E = z (x' + y) + y' = x' z + y z + y'$

Then $E = x' z + y z + y' = x' z (y + y') + y z (x + x') + y'(x + x') (z + z')$

$= x' y z + x' y' z + x y z + x' y z + x y' z + x y' z' + x' y' z + x' y' z'$

$= x y z + x y' z + x y' z' + x' y z + x' y' z + x' y' z'$

Example 8 (2018):- Write the Boolean expression E (x, y, z) first as a sum of products and in complete sum of product form.

Solution :

(a) Let E(x, y, z) is $E = y (x + y z)'$
(b) E(x, y, z) is $E =.x (x y + y' + x' y)$

Solution for (a) :- $E = y (x' (y z)') = y x' (y' + z') = y x' y' + x' y z' = x' y z'$

Which is in required complete sum of product form.

Solution for (b) :- First we have $E = x\,x\,y + x\,y' + x\,x'\,y = x\,y + x\,y'$

Then $E = x\,y\,(z + z') + x\,y'(z + z') = xyz + xyz' + x\,y'\,z + x\,y'\,z'$

Example 9 (2018):- Consider Boolean algebra D_{210} . The divisor of 210. Also find the number of sub-algebra of D_{210} .

Solution : A sub-algebra of D_{210} must

(i) There can be only one two elements sub-algebra which consists of the upper bound 210 and lower bound 1 i.e, {1, 210 }

(ii) Since D_{210} consists sixteen elements . The only sixteen elements sub-algebra is D_{210} itself.

(iii) Any four elements sub-algebra is of the form { 1, x, x′, 210 . That is consists of upper bound and lower bound and a non bound element together with its complement. These are fourteen non bound elements in D_{210} and so these are 14/2 = 7 pairs { x, x′ }. Thus D_{210} has seven four elements sub-algebras.

(iv) Any eight element sub-algebra say S will itself contain three atoms S_1, S_2, S_3. We can choose S_1 and S_2 to be any two of the four atoms of D_{210} and then S_3 must be the product of the other two atoms. For examples :
We can let $S_1 = 2$, $S_2 = 3$, $S_3 = 5 . 7 = 35$ (which determines the sub-algebra { 1, 2, 3, 35, 6, 70, 105, 210 }
or we let $S_1 = 5$, $S_2 = 7$, $S_3 = 2 . 3 = 6$ (which determines the sub-algebra { 1, 5, 6, 7, 30, 35, 42, 210 }
There are ${}^4C_2 = 6$ ways to choose S_1 and S_2 from the four atoms { 2, 3, 5, 7 } of D_{210} and so D_{210} has six ‘ eight element ’ sub-algebra.
Accordingly D_{210} has 1 + 1 + 7 + 6 = 15 sub-algebras.

Also the divisor of D_{210} are 1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105 and 210.

Example 10 (2017, 2019):- For x, y belonging to a Boolean algebra B prove that

(i) $x \wedge y' = 0$ (iii) $x' \vee y = 1$

(ii) $x \vee y = y$ (iv) $x \wedge y = x$

Solution : by a theorem (ii) and (iv) are equivalent

Now we show that (ii) and (iii) are equivalent

For this, let (ii) holds then $x' \vee y = x' \vee (x \vee y) = (x' \vee x) \vee y = 1 \vee y = 1$

Now let (iii) holds then $x \vee y = 1 \wedge (x \vee y) = (x' \vee y) \cap (x \vee y) = (x' \cap x) \vee y = 0 \vee y = y$

Thus (ii) and (iii) are equivalent

For this,

Let (iii) holds. By De – Morgan's law and involution

$0 = 1' = (x' \vee y') = x'' \cap y' = x \cap y'$

Conversely : If (i) holds then $1 = 0' = (x \cap y')' = x' \cup y'' = x' \vee y$

Thus (i) and (iii) are equivalent.

That is all four are equivalent.